

**A New Approach to
Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -Controller Synthesis
Using Gradient-Based Parameter
Optimization Methods**

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Abstract

In the past few years, the *Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -Control Problem* has been the object of much research interest, since it allows the incorporation of robust stability into the LQG-framework. The general *Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -Design Problem* has yet to be solved analytically. Numerous schemes have considered upper bounds for the \mathcal{H}_2 -performance criterion and/or imposed restrictive constraints on the class of systems under investigation. Furthermore, many modern control applications rely on dynamic models obtained from finite-element analysis and thus involve high-order plant models. Hence the capability to design low-order (fixed-order) controllers is of great importance. In this research a new design method was developed, that optimizes the exact \mathcal{H}_2 -norm of a certain subsystem subject to robust stability in terms of \mathcal{H}_∞ -constraints and a minimal number of system assumptions. The derived algorithm is based on a differentiable scalar time-domain penalty function to represent the \mathcal{H}_∞ -constraints in the overall optimization. The scheme is capable of handling multiple plant conditions and hence multiple performance criteria and \mathcal{H}_∞ -constraints, and incorporates additional constraints such as fixed-order and/or fixed-structure controllers. The defined penalty function is applicable to any constraint that is expressible in form of a real symmetric matrix-inequality.

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GLOSSARY

Basic Notation:

j	: Square root of -1 .
$k!$: Factorial of a positive integer k .
z_r, z_i	: Real and imaginary part of a complex number z .
$ z $: Absolute value of a (complex) number z .
\in	: Element of a set, e.g., $x \in L_2$.
$:=$: Equivalence definition, i.e. the state-space realization of a linear system $G := (A, B, C, D)$.
\mapsto	: Mapping from one space to another, e.g. the mapping of L_2 -signals $w(s) \in R^{n_w}$ to signals $z(s) \in R^{n_z}$ defined by the transfer function $G(s)$ of a linear system, $G(s) : w(s) \mapsto z(s)$.

Vector and Matrix Notation:

$\dim(x)$: Dimension of a vector x .
$\mathcal{E}(x)$: Expected value of x .
I	: Identity matrix.
$\text{vec}(M)$: Vector representation of a matrix M (see appendix A).
$\text{diag}(M)$: Diagonal matrix with the main diagonal of M and zero entries everywhere else.
$\text{Trace}(M)$: Trace operator.
$ M $: Determinant of a square matrix M .
M^T	: Transpose of M .
M^{-1}	: Inverse of M .

M^+	: Generalized Moore Penrose inverse of a matrix M .
e^M	: Matrix exponential.
$M > 0$ ($M \geq 0$)	: The symmetric matrix M is positive definite (semidefinite).
$\mathcal{R}(M)$: Range space of the matrix M .
$\mathcal{N}(M)$: Null space of the matrix M .
$\lambda_k(M)$, $\bar{\lambda}(M)$: k^{th} and maximal eigenvalue of M , respectively.
$\sigma_k(M)$, $\bar{\sigma}(M)$: k^{th} and maximum singular value of M , respectively.

Gradient-Related Notation:

$df(K, dK)$: Variation of a scalar cost function $f(K)$ due to a matrix variation dK (see appendix A).
$dM(K, dK)$: Matrix equivalent of $df(K, dK)$ for a matrix-valued function of a matrix K (see appendix A).
$\frac{\delta f}{\delta x}$, $\frac{\delta f}{\delta K}$: Gradient of a function with respect to a scalar or vector x and matrix K respectively.

Norm-Related Notation:

$\ x\ _2$: The L_2 -norm of a (continuous-time) vector signal $x(t)$: $\ x\ _2^2 = \int_0^\infty x^T(t)x(t)dt$.
L_2	: Space of (continuous-time) vector-valued signals with finite L_2 -norm (square integrable).
$\ M\ _F$: Froebenius norm of a (real) matrix: $\ M\ _F = [\text{Trace}(MM^T)]^{\frac{1}{2}}$.
$G(j\omega)$: $G(s) \mid_{s=j\omega}$.
$\ G\ _2$: The \mathcal{H}_2 -norm of a transfer function $G(s)$.
$\ G\ _\infty$: The \mathcal{H}_∞ -norm of a transfer function $G(s)$.

Some Specific Notation Adopted in this Work:

- $A^i, B_k^i, C_k^i, D_{k,l}^i$: State-space matrices of the i^{th} plant condition with $k, l = 1, 2, 3$.
- $ARI_{C,OF}^i(C_0, X^i, \gamma^i)$: Algebraic Riccati Inequality representing the i^{th} \mathcal{H}_∞ -constraint in the general continuous-time output-feedback multi-plant case (see chapter 5).
- $ARI_{C,SF}(C_0, X, \gamma)$: Algebraic Riccati Inequality representing the \mathcal{H}_∞ -constraint in the continuous-time full state-feedback single-plant case (see chapter 6).
- $ARI_{D,SF}(C_0, X, \gamma)$: Algebraic Riccati Inequality (Linear Matrix Inequality) representing the \mathcal{H}_∞ -constraint in the discrete-time full state-feedback single-plant case (see appendix D).

Acronyms:

- ARE : Algebraic Riccati Equation.
- ARI : Algebraic Riccati Inequality.
- DGKF : Doyle, Glover, Khargonekar and Francis.
- LMI : Linear Matrix Inequality.
- LQ : Linear Quadratic.
- LQR : Linear Quadratic Regulator.
- LQG : Linear Quadratic Gaussian.
- MIMO : Multi Input/Multi Output.
- RMS : Root-Mean-Square.
- SISO : Single Input/Single Output.

Chapter 1

INTRODUCTION

1.1 Why Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -Control: Motivation

Control theorists have developed a formidable framework for the analysis and design of control systems for linear time-invariant (LTI) systems. One of the remaining problems, however, is that of uncertain and disturbed systems. Mathematical modeling of a physical system provides the basis for the design of feedback controllers. In order to obtain a simple model of the dynamic process, compromises have to be made between the fidelity of the system model and the complexity of its model description. This process invariably implies modeling errors and hence uncertainties in the model description. Such uncertainties can include neglected high-frequency dynamics, variables that might change during the course of operation, or parameters that are just unknown but bounded within a certain domain. Additionally, in many cases measurements and actuator signals will be noise corrupted. All of these uncertainties have to be taken into account when designing a controller.

In many modern engineering applications the use of lightweight materials has become a necessity to conserve energy, fuel or other resources. This is true especially in fields such as aeronautics and astronautics or robotics. Modeling of such plants typically relies on finite element techniques and hence involves high-order plant models. In order to arrive at a practically implementable controller, the chosen design method must be able to provide the capability to design fixed-order controllers (that is, controllers with a prespecified low order regardless of the plant-order). Due to physical limitations many applications also require the design of structurally constrained controllers. Furthermore, few plants will have the same system model over the whole range of operation. To avoid techniques such as gain-scheduling or adaptive control schemes, it is desirable to design a single controller that takes into account

various operating conditions. In short, a successful controller design paradigm must take into account model uncertainties and disturbances and be able to accommodate requirements such as fixed-order/fixed-structure controllers.

At this point we have to clearly differentiate between two design objectives in the control-law synthesis. First and foremost the controller must provide robust stability. That is, the closed-loop system must be asymptotically stable for all considered uncertainties and disturbances. Once the closed-loop stability has been satisfied, we may require that additional performance specifications are met as well. Ideally one would like to design a controller that provides acceptable performance (in some sense) and guarantees stability for all uncertainties. However, this most general problem of *Robust Performance with Robust Stability* is theoretically hard to tackle, even for very simple performance measures, and is a matter of ongoing research. Instead, the problem considered in this work is the problem of *Nominal Performance with Robust Stability*. That is, the performance measure is optimized for the nominal plant while stability is guaranteed for a set of bounded uncertainties. Using recent results on robust performance, this design scheme will be related to the general problem of *Robust Performance with Robust Stability*. In this introduction we will make frequent use of system and signal norms such as \mathcal{H}_2 , \mathcal{H}_∞ and L_2 . These norms are defined in chapter 2. The reader unfamiliar with these norm concepts is urged to consult chapter 2.

\mathcal{H}_2 -theory has a long history. The first step in this development was the formulation of quadratic performance indices for deterministic systems. The *Linear Quadratic Regulator* (LQR) design philosophy was the outcome of this research effort. This framework then was extended to include noise corrupted systems where measurement and system noises were modeled as white-noise processes. This approach extended the concept of LQ-performance to systems with noise corruption and resulted in the *Linear Quadratic Gaussian* (LQG) framework. This design method can in turn be formulated in terms of an \mathcal{H}_2 -optimal control problem. Stochastic disturbances with distributions other than white noise are easily incorporated into this framework by the use of shaping filters. Thus, for LQ-type performance problems in noise corrupted systems with stochastic disturbances of known distribution, \mathcal{H}_2 -optimal control is the tool of choice. Despite their nice interpretation in terms of stochastic disturbance rejection and LQ-performance, \mathcal{H}_2 -optimal controllers have

one important drawback. The plant model is assumed to be known exactly! For this nominal plant \mathcal{H}_2 -optimal controllers guarantee well documented stability margins. If the plant is perturbed due to plant parameter uncertainties, however, \mathcal{H}_2 -optimal controllers may no longer guarantee closed-loop stability. This philosophy is, in its very essence, a performance-oriented design framework that was not intended to solve the stability problem inherent to uncertain systems. Future research may develop a scheme in which the proper choice of weighting matrices in the quadratic cost function may provide a tool to define robust stability even for this method. Research, however, has not progressed to this point yet. The \mathcal{H}_2 -norm of linear time-invariant systems is not applicable to the robust stability problem in a small gain framework because the \mathcal{H}_2 -norm of transfer functions is not submultiplicative and has no interpretation as a transfer function gain. In short, \mathcal{H}_2 provides an excellent framework for performance considerations and white-noise disturbance rejection as developed in the LQR and LQG design philosophies. In a more realistic concept, however, mathematical models are necessarily uncertain and the resulting stability problem is not solvable with this approach alone.

Starting with the early works of Hurwitz, Schur and Lure, stability theory for linear time-invariant systems has progressed to modern robust stability approaches such as Kharitonov-type theorems, parameter-dependent Lyapunov functions, the concept of stability radii of system matrices and criteria based on singular values such as the Bounded Real Lemma and the Small Gain Theorem. \mathcal{H}_∞ -analysis and synthesis are based on the Small Gain Theorem and form an effective method to account for norm-bounded plant uncertainties for the design of controllers that guarantee robust stability. The types of uncertainties that can be accommodated include static parametric uncertainties as well as dynamic uncertainties such as unmodeled high-frequency plant, actuator and sensor dynamics. The perturbations may be real, complex, scalar or matrix-valued and hence encompass a wide variety of practical design problems. However, the \mathcal{H}_∞ -framework in its present form is concerned with the design of linear controllers for linear plants. Hence many problems (in particular problems involving nonlinear uncertainties) cannot be tackled with this method. Extensions of the \mathcal{H}_∞ -scheme to these kinds of problems are currently under investigation.

The \mathcal{H}_∞ -norm of a system is the maximum singular value of the transfer function over all frequencies and can be interpreted as the maximum *gain* of a transfer function for all L_2 -bounded input signals, or as the maximum ratio of the L_2 -norm of the system output and the L_2 -norm of the system input (see chapter 2). The interpretation of the \mathcal{H}_∞ -norm as the worst-case *gain* and the fact that the \mathcal{H}_∞ -norm is submultiplicative make this norm an appropriate mathematical tool for a robust stability criterion within the framework of the Small Gain Theorem. Although the Small Gain Theorem provides only a sufficient criterion for robust stability, the \mathcal{H}_∞ -framework has become increasingly popular in the last decade. This research effort is justified by the extendability of \mathcal{H}_∞ -methods to the *Upper Bound μ -Problem* and the ease of computation in contrast to other methods. μ -theory gives necessary and sufficient conditions for robust stability. Computationally, however, this problem has not yet been solved. By convention the term “ \mathcal{H}_∞ -design problem” refers to the design of suboptimal \mathcal{H}_∞ -controllers. Suboptimal controllers satisfy a prespecified stability bound in contrast to optimal \mathcal{H}_∞ -controllers that actually minimize the robust stability measure. \mathcal{H}_∞ -optimal controllers, however, are often undesirable due to the occurrence of high gain and large controller bandwidth. Most suboptimal \mathcal{H}_∞ -controllers do not exhibit these disadvantages. For a large class of problems suboptimal \mathcal{H}_∞ -controllers are easily computed via the solutions of two Riccati equations with an associated coupling condition. Unlike for \mathcal{H}_2 , the separation principle is not valid in the case of \mathcal{H}_∞ . In summary, \mathcal{H}_∞ provides a suitable framework for the incorporation of uncertainty into a controller design concept for robust stability.

On the other hand, possible performance considerations in this framework alone must necessarily be of limited scope. Let us, for the moment, consider a Multiple Input/Multiple Output (MIMO) system with m inputs and m outputs. For such a system, the \mathcal{H}_∞ -norm depicts, by definition, only one point of the maximum singular value function (as a function of frequency) which is one out of m possible singular value functions. Considering the limited information that the \mathcal{H}_∞ -norm provides about the overall internal structure of a system, a controller design approach based on \mathcal{H}_∞ will not be able to incorporate LQ performance measures in a way \mathcal{H}_2 does. Within the framework of μ -theory some approaches in this direction have been formulated nonetheless. The performance measures there are only of the maximum-singular-value-type and must not be confused with \mathcal{H}_2 -performance specifications.

\mathcal{H}_∞ is an important tool for robust stability, but it is useful only as a robust stability constraint in an overall performance-oriented scheme that takes into account all the other design specifications that a comprehensive controller design paradigm has to satisfy.

It is obvious at this point that the combination of \mathcal{H}_2 and \mathcal{H}_∞ into a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -framework incorporates the advantages of both approaches. It allows the formulation of performance specifications in a more realistic fashion, namely for uncertain systems. It eliminates the stability problems inherent to the \mathcal{H}_2 -philosophy and adds the performance aspect in terms of \mathcal{H}_2 to the \mathcal{H}_∞ -framework. The resulting approach for the problem of *\mathcal{H}_2 -Performance with \mathcal{H}_∞ -Robust Stability* is a relatively new and promising approach to combine these design specifications into one comprehensive design concept for a large class of real world problems. These advantages sparked an enormous research effort in this field in the last few years.

Within the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design methodology one has to differentiate between *Nominal \mathcal{H}_2 -Performance with \mathcal{H}_∞ -Robust Stability* and *Robust \mathcal{H}_2 -Performance with \mathcal{H}_∞ -Robust Stability*. In the first approach the \mathcal{H}_2 -norm of the nominal transfer is minimized. This approach has the advantage that the \mathcal{H}_2 -problem and the \mathcal{H}_∞ -problem can be considered separately. The latter approach seeks to minimize the corresponding \mathcal{H}_2 -norm for the perturbed plant. This problem is still unsolved. By convention the expression “mixed $\mathcal{H}_2/\mathcal{H}_\infty$ ” will refer to the problem of *Nominal \mathcal{H}_2 -Performance with \mathcal{H}_∞ -Robust Stability*. Within this methodology two different directions have evolved. In most cases there will be a conflict between the \mathcal{H}_2 - and the \mathcal{H}_∞ -objectives. This means that tighter robust stability bounds will lead to deteriorating \mathcal{H}_2 -performance and vice versa. This problem is referred to as the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design problem. The goal in this philosophy is to find a controller that minimizes the \mathcal{H}_2 -norm of a given transfer function, subject to an \mathcal{H}_∞ -constraint on another (possibly different) linear system. This approach is drastically different from the *Simultaneous $\mathcal{H}_2/\mathcal{H}_\infty$ -Optimal Control Problem* ([91], [96]), where one seeks a controller that minimizes the \mathcal{H}_2 -norm of a given transfer function while *simultaneously* satisfying the desired \mathcal{H}_∞ -constraint. This type of problem will be solvable only for special cases. The difference between these approaches is that the simultaneous $\mathcal{H}_2/\mathcal{H}_\infty$ -optimal controller will always be \mathcal{H}_2 -optimal while this may not be the case for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design approach.

The problem considered in this research is that of *Nominal \mathcal{H}_2 -Performance with \mathcal{H}_∞ -Robust Stability* and can now be posed as follows. Under a minimal number of system assumptions, the problem is to find a controller of fixed order/fixed structure that minimizes the \mathcal{H}_2 -norm of a given transfer function, subject to an \mathcal{H}_∞ -bound on another linear system. Finally, the design concept should be easily extendable to the multi-plant case.

1.2 Related Literature

In general this problem formulation involves two subproblems. The \mathcal{H}_2 -control problem has been treated extensively in the last two decades and will be reviewed only briefly. One of the important contributions of modern \mathcal{H}_2 -research is the interpretation and setting of \mathcal{H}_2 in the frequency-domain ([25], [111]). Most recent advances can be found in [18], where a generalized parametrization for \mathcal{H}_2 -optimal controllers has been derived. In most present schemes, the order and structure of the controller may not be arbitrarily pre-assigned. A recent approach using homotopy methods can be found in [19]. One method for the gradient-based design of constrained \mathcal{H}_2 -controllers was developed in [64]. In particular, the method developed there does not require an initially stabilizing controller, it allows the design of structurally constrained controllers and incorporates the design capability for multiple plants. For further discussion of nominal \mathcal{H}_2 -related problems the reader is referred to standard publications such as [66]. Robust \mathcal{H}_2 -design on the other hand is a matter of present research and some results are slowly forthcoming. As already mentioned, the general problem of robust \mathcal{H}_2 is not solved yet. All approaches in this direction represent upper bounds for this robust performance; noteworthy in this respect is the work by Stoorvogel ([114], [117]). Following a different approach, so-called *guaranteed \mathcal{H}_2 -cost controllers* have been investigated in [39] and [83].

\mathcal{H}_∞ -theory on the other hand has a rather short history. Although the Small Gain Theorem was introduced by Zames in 1966 ([136]), it took another fifteen years until the same author applied this concept to the disturbance attenuation problem for deterministic uncertainties. Zames's 1981 seminal paper ([137]) has to be considered the beginning of \mathcal{H}_∞ -theory. Using the Youla (Q-) parametrization for all stabilizing controllers, it was shown that the \mathcal{H}_∞ -design problem is in general infinite-dimensional ([12], [66]). Based on a formulation of the \mathcal{H}_∞ -problem in terms of a Hankel ap-

proximation problem, subsequent research derived various frequency-domain solution methods for the one, two and general four-block problems ([23], [27], [28], [29], [44], [46], [132], [138]). Computationally these methods were cumbersome and did not attract much attention in practical applications. The celebrated 1989 paper by Doyle et. al. ([24], see also [43]) solved the *regular* suboptimal \mathcal{H}_∞ -design problem and presented a Two-Riccati \mathcal{H}_∞ -Solution (also termed the DGKF-equations). Furthermore, in this paper it was shown that a suboptimal \mathcal{H}_∞ -controller – if such a controller exists – will be of the same order as the plant. The resulting controller is termed the *Central Controller*. For properties of the *Central Controller* such as pole-zero cancellation as well as lifting techniques for some of the imposed system assumptions, the reader is referred to [97] and [108]. Additional information on computational issues can be found in [32] and [33]. The ease of computation and the fact that this controller is of finite dimensions made this method a powerful tool. In [70] some properties of the *Non-Central Controller* are examined. Further development illustrated the intimate relationship between the ARE-based \mathcal{H}_∞ -approach and certain game-theoretical problems ([1], [2], [4], [5], [51], [73], [78], [87], [103], [120], [125], [126], [127], [129]). Connections between Riccati equation approaches to the \mathcal{H}_∞ -problem, game-theory and quadratic stabilizability have been examined by Petersen ([77], [78], [79], [80], [81], [82], [84]).

The *singular* \mathcal{H}_∞ -problem was solved by Stoorvogel ([113], [132], [116], [119]) in terms of quadratic matrix inequalities and various rank conditions. A recent extension of this approach to the reduced-order case can be found in [118]. Although computationally not as attractive as the method in [24], this approach removed some very restrictive system assumptions imposed in [24]. One of the remaining assumptions was a rather restrictive constraint on the system zeros. Removal of this assumption requires either the use of perturbation techniques ([97]) or a reformulation of the problem in terms of Algebraic Riccati Inequalities (ARI's) ([48], [98], [99]). The significance of matrix inequalities in systems theory was recognized very early and has sparked renewed interest in this technique over the last few years ([15]). The characterization of \mathcal{H}_∞ -bounds in terms of matrix inequalities can be traced back to a paper by Willems ([134], see also [128]). This idea was utilized in a paper by Zhou and Khargonekar ([141]) and forms the basis for the most advanced \mathcal{H}_∞ -design methods available at this time. A characterization of all suboptimal \mathcal{H}_∞ -controllers

has been developed by Gahinet ([35], [36], [37]) as well as by Iwasaki and Skelton ([52], [53]). This method requires the solution of two convex matrix inequalities subject to a rank inequality. This framework has to be considered more general than ARE-based methods as it does not require observability or controllability of the considered plant, only detectability and stabilizability are needed. These assumptions are required for the existence of a stabilizing controller and hence do not represent a loss of generality. Parallel to this work Geromel et. al. and Peres developed a similar technique that resulted in a convex parametrization of all full state-feedback suboptimal \mathcal{H}_∞ -controllers ([38], [40], [41], [76]). Theoretically these results give a complete characterization of all full state-feedback and reduced-order output-feedback \mathcal{H}_∞ -controllers. The computation of solutions for matrix inequalities, however, remains still a matter of current research ([14], [15], [100], [103]). The number of publications related to the robust stability problem is extensive. The above citations represent only the most important recent papers. For a comprehensive list of publications in the last few years the reader is referred to [21].

A very general framework for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design problem has been developed by Ridgely ([89]) as well as Steinbuch and Bosgra ([112]). The approach utilizes a set of Lagrange multipliers to append the \mathcal{H}_∞ -constraint in terms of an ARE to the \mathcal{H}_2 -performance cost. Corresponding gradients give necessary conditions for the derivation of a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -controller. Gradient-based methods are used to compute the controller. The approach incorporates many important features such as fixed-order/fixed-structure controllers. However, in addition to a set of rather restrictive system assumptions, this approach requires an initially stabilizing controller that satisfies the desired \mathcal{H}_∞ -bound. Also, due to the \mathcal{H}_∞ -characterization by an ARE, this problem formulation requires the corresponding \mathcal{H}_∞ -problem to be *regular* (*non-singular*, [24]).

A special class of systems has been considered by Bernstein and Haddad. For these systems the same outputs for the \mathcal{H}_2 and \mathcal{H}_∞ -criteria, but different disturbance input channels are assumed. For these systems a scheme has been developed by Bernstein and Haddad ([6], [7], [8]). In this approach an upper bound for the corresponding \mathcal{H}_2 -norm is minimized while the specified \mathcal{H}_∞ -constraint is satisfied. The computation of fixed-order controllers in this scheme involves the solution of up to six coupled Riccati equations. Only homotopy methods are available to solve

such a computationally challenging task ([9]). The dual problem has been solved in [26] and [140]. Subsequently it has been shown in [135] that the conditions derived in [6] and [26] are necessary and sufficient. For the case of full state-feedback, the above idea can be formulated as a convex optimization problem using the controller parametrization introduced in [38] and a constraint in form of a convex matrix inequality. This particular problem has been investigated by Rotea, Khargonekar and coworkers ([56], [58], [91], [93], [94]).

Yet another approach to mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design is based on the γ -Entropy of a system ([68], [69], [71]). This function is finite only if the \mathcal{H}_∞ -constraint is satisfied. In this case the negative function value represents an upper bound for the \mathcal{H}_2 -norm of this system. Unfortunately this formulation is applicable only if the \mathcal{H}_2 -objective and the \mathcal{H}_∞ -constraint are related to the same transfer function.

1.3 Outline of the Report

This introduction is followed by preliminary definitions and results in chapter 2. There relevant norms are defined, uncertain systems and their representation as well as the Small Gain Theorem and various types of \mathcal{H}_∞ -bound characterizations are presented. In chapter 3 the actual problem definition is stated along with system assumptions and state-space representations of the considered plants. Chapter 4 provides an introduction to matrix inequalities and their relevance in control systems theory. A new differentiable scalar cost function is defined that represents the corresponding \mathcal{H}_∞ -constraint in the overall optimization scheme. Some properties and possible applications of this cost function are discussed. A new approach to the multi-plant \mathcal{H}_∞ -design problem based on this new cost function is then formulated in chapter 5. The related optimization can be performed either as a constrained optimization or via a penalty/barrier function approach. A discussion of these algorithms and some numerical advice are included in this chapter. An extension of this scheme naturally leads to the formulation of the multi-plant $\mathcal{H}_2/\mathcal{H}_\infty$ -design problem in terms of the defined scalar cost function. This design problem is considered in chapter 6. The state-feedback single-plant case with an upper bound for the corresponding \mathcal{H}_2 -cost and identical disturbance inputs for the \mathcal{H}_2 and \mathcal{H}_∞ -criteria as treated in [58] can be found in this chapter as well. This chapter is followed by concluding remarks in chapter 7 and some comments on possible extensions of the presented scheme as

well as an outline of potential future research directions in chapter 8. In appendix A auxiliary matrix results that are required for various proofs are presented along with the theoretical framework necessary for the computation of gradients for all the cost functions that are defined throughout this report. Explicit gradient expressions for the relevant cost functions can be found in appendix B. In appendices C and D it is shown that the scalar cost function can also be used for \mathcal{H}_∞ -constrained optimization problems where the performance measures are not \mathcal{H}_2 . In appendix C the *Minimum Gain Problem* subject to an \mathcal{H}_∞ -constraint for the full state-feedback case in the continuous-time domain is examined. Appendix D shows the applicability of the presented scheme to \mathcal{H}_∞ -problems in the discrete-time domain. As an \mathcal{H}_∞ -constraint in the discrete-time domain can be represented via an ARI, the proposed cost function can also be used to impose the \mathcal{H}_∞ -constraint. The objective is the nominal L_1 -norm of the closed-loop A -matrix which has interpretations in terms of time-domain constraints on the state and the control vectors.

Except for the results presented in appendix D, the formulation and treatment of the \mathcal{H}_∞ -design problem and the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -problem are cast completely in the continuous-time domain. An exception to this rule was made only to demonstrate the applicability of this scheme to discrete-time problems.

In appendix E the accompanying MATLAB \mathcal{H}_∞ -design software based on the results in chapter 5 is described in details. Usage of the software is illustrated by examples and possible causes for non-convergence of the algorithm are also discussed. Although the same algorithm has been utilized for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design examples, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design software still requires the adjustment of parameters in intermediate steps of the algorithm. Hence, for this reason it is not included in this appendix. Note that in the appendix E a different notation has been adopted for the description of the open-loop systems. The notation follows closely the standard notation used in the literature for the pure \mathcal{H}_∞ -problem and it differs from the notation used in the body of this report.

The following notational convention is used throughout this report. In general signal dependency on time will be shown explicitly while a possible frequency dependency is omitted. The context will identify whether the time-domain or the frequency-domain is considered. Furthermore, for linear time-invariant systems the

term “asymptotic stability” refers to stability over an infinite time-horizon and hence exponential stability of the system under consideration ([16]). Similarly, a matrix M is asymptotically stable if the real parts of every eigenvalue of M is negative. Conversely, a matrix M is “antistable” if $-M$ is asymptotically stable. The term “mode” is equivalent to an eigenvalue of a matrix. Also, a linear time-invariant system is minimal if it is both observable and controllable. For brevity the term “transfer function” is generally used to refer to transfer function matrices in the case of multi-input/multi-output systems. The remaining abbreviations can be found either in the glossary or are defined in the body of this report and are fairly standard.

1.4 Contributions

1. A new scalar time-domain cost function is defined that allows the representation of \mathcal{H}_∞ -constraints in terms of a scalar constraint. The defined cost function is continuous and differentiable. Explicit gradient expressions are provided and hence standard nonlinear gradient-based software may be applied to solve the suboptimal multi-plant \mathcal{H}_∞ -design problem. An iterative scheme is presented to numerically solve the associated optimization problem. Furthermore the possible extension to a multi-plant \mathcal{H}_∞ -optimal controller design method is outlined. The considered plants need not be of the same order and for each plant a different set of input/output vectors may be defined that will be subject to the \mathcal{H}_∞ -constraints. The developed scheme includes features such as multiple plants, multiple \mathcal{H}_∞ -constraints, full state-feedback, strictly proper and proper controllers with fixed-order and/or structure. The initial controller guess is not required to be stabilizing. Furthermore the system assumptions are the least restrictive.
2. Based on the proposed cost function and hence the scalar representation of \mathcal{H}_∞ -constraints a new approach for the multi-plant problem of *Nominal \mathcal{H}_2 -Performance with \mathcal{H}_∞ -Robust Stability* is formulated. For multiple plants the \mathcal{H}_2 -criterion is a weighted sum of the individual \mathcal{H}_2 -norms of each plant condition. Due to the differentiability of the \mathcal{H}_2 -criterion the numerical treatment is equivalent to that of the pure \mathcal{H}_∞ -design problem including the same features and capabilities as described above.

3. The scheme is applicable to problems where other performance criteria are desired and to \mathcal{H}_∞ -constrained problems in the discrete-time domain. The *Minimum Gain Problem* subject to an \mathcal{H}_∞ -constraint for the state-feedback case in the continuous-time domain and the \mathcal{H}_∞ -constrained control problem with time-domain constraints in the discrete-time domain have been shown to be solvable with the presented approach.
4. The proposed cost function is applicable to enforce any kind of matrix constraint in the form of a scalar inequality constraint as long as the matrix constraint under consideration is expressible in terms of a real symmetric differentiable matrix inequality. This property allows the application of the developed scheme to many other constrained optimization problems. In particular it is shown that the proposed cost function is convex if the underlying matrix inequality is convex and hence convexity of the original constraint is preserved.

Chapter 2

PRELIMINARIES

2.1 Norm Definitions

In the following $G := (A, B, C, D)$ will denote the state-space realization of a LTI system G with $w(t)$ as input and $z(t)$ as output.

$$G : \begin{cases} \dot{x}(t) &= Ax(t) + Bw(t), \quad x(0) = x_0 \\ z(t) &= Cx(t) + Dw(t) \end{cases} \quad (2.1)$$

The corresponding transfer function from $w(s)$ to $z(s)$ in the frequency-domain is denoted by $G(s) = C(sI - A)^{-1}B + D$ for $x_0 = 0$. One can define a multitude of norms for LTI systems. Among these, the \mathcal{H}_2 and \mathcal{H}_∞ -norms have become the most widely used system norms due to their nice mathematical properties and intuitive interpretations. Note that the \mathcal{H}_2 -norm is defined only for strictly proper asymptotically stable systems, the \mathcal{H}_2 -norm of proper or unstable systems is infinite.

Definition 2.1.1

Consider a strictly proper asymptotically stable LTI system $G := (A, B, C, 0)$ with corresponding transfer function $G(s)$, then the \mathcal{H}_2 -norm $\|G\|_2$ is defined as follows.

1.

$$\|G\|_2 = \sqrt{\text{Trace}(\int_0^\infty C e^{At} B B^T e^{A^T t} C^T dt)} \quad (2.2)$$

$$= \sqrt{\text{Trace}(\int_0^\infty B^T e^{A^T t} C^T C e^{At} B dt)} \quad (2.3)$$

2. Let w be white noise signals with unit power spectral density $S_{ww}(j\omega) = I$, then

$$\|G\|_2 = \sqrt{\text{Trace}[\frac{1}{2\pi} \int_{-\infty}^\infty G(j\omega) G^T(-j\omega) d\omega]} \quad (2.4)$$

$$= \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \|G(j\omega)\|_F^2 d\omega} \quad (2.5)$$

$$= \lim_{t_f \rightarrow \infty} \sqrt{\mathcal{E}[z^T(t_f)z(t_f)]}. \quad (2.6)$$

The characterization of $\|G\|_2$ in terms of (2.2) and (2.3) gives rise to the familiar computation of this norm in terms of the controllability or observability grammians. Equations (2.4), (2.5) and (2.6) illustrate the interpretation of the \mathcal{H}_2 -norm in the frequency-domain in terms of stochastic white-noise signals as driving inputs. For this type of disturbance input, the \mathcal{H}_2 -norm is an appropriate measure for the energy (RMS value) of the output. In particular (2.6) shows that $\|G\|_2$ can be computed via a finite-time cost function $\sqrt{\mathcal{E}[z^T(t_f)z(t_f)]}$ in the limit as $t_f \rightarrow \infty$. In a practical implementation one need not go to the actual limit $t_f \rightarrow \infty$. Depending on the eigenvalues of the system, a large but finite t_f would adequately approximate the true \mathcal{H}_2 -norm of the system. This fact has been utilized for a very general design algorithm in [64].

The \mathcal{H}_∞ -norm, on the other hand, is a gain norm. As with the \mathcal{H}_2 -norm, there are interpretations in the time and frequency-domains.

Definition 2.1.2

Consider an asymptotically stable LTI system $G := (A, B, C, D)$ with the corresponding transfer function $G(s)$ and $x_0 = 0$, then the \mathcal{H}_∞ -norm $\|G\|_\infty$ is defined as follows.

$$\|G\|_\infty = \sup_{w, \|w\|_2=1} \lim_{t_f \rightarrow \infty} \sqrt{\frac{\int_0^{t_f} z^T(t)z(t)dt}{\int_0^{t_f} w^T(t)w(t)dt}} \quad (2.7)$$

$$= \sup_{w, \|w\|_2=1} \frac{\|z\|_2}{\|w\|_2} \quad (2.8)$$

$$= \sup_{\omega} \sqrt{\bar{\lambda}[G^T(-j\omega)G(j\omega)]} \quad (2.9)$$

$$= \sup_{\omega} \bar{\sigma}[G(j\omega)], \quad (2.10)$$

where $\bar{\lambda}(\cdot)$ and $\bar{\sigma}(\cdot)$ are the maximum eigenvalue and maximum singular value of the arguments respectively.

Physically this norm is the worst-case ratio of output energy to input energy for input signals with bounded energy. Very interesting in this respect is the fact that periodic

signals have unbounded energy and hence are not included in the above time-domain definition. That is, the supremum in equations (2.7) and (2.8) will not be achieved for the considered class of disturbance signals $w(t) \in L_2$. In general we have

$$\int_0^{t_f} z^T(t)z(t)dt \leq \|G\|_\infty^2 \int_0^{t_f} w^T(t)w(t)dt \quad x_0 = 0, \quad \forall w(t) \in L_2, \quad \forall t_f > 0. \quad (2.11)$$

Reference [26] contains a more in-depth discussion along these lines. The frequency-domain definition shows that this norm is the worst-case gain over all frequencies. The class L_2 of disturbance signals for which the \mathcal{H}_∞ -norm is defined, makes the combination of \mathcal{H}_2 and \mathcal{H}_∞ so attractive. General L_2 -disturbances do not have nice interpretations in the \mathcal{H}_2 -framework, stochastic noise signals on the other hand have no relevance in \mathcal{H}_∞ -theory. In addition to the usual properties of operator norms, the \mathcal{H}_∞ -norm is submultiplicative. That is,

$$\|GH\|_\infty \leq \|G\|_\infty \|H\|_\infty \quad (2.12)$$

for two transfer functions $G(s)$ and $H(s)$. Operator norms that satisfy this inequality are also called generalized operator norms. The above inequality does not hold in general for the \mathcal{H}_2 -norm. Submultiplicativity is the key feature of the \mathcal{H}_∞ -norm that allows its application to the robust stability problem via the Small Gain Theorem.

2.2 Equivalent Representations of \mathcal{H}_∞ -Constraints for LTI Systems

Even though there is no analytical “one-step” solution for the most general \mathcal{H}_2 -design problem, efficient solutions for this design strategy with structurally constrained and fixed-order controllers have been developed ([64]). The true problem associated with the general $\mathcal{H}_2/\mathcal{H}_\infty$ -design task is that of the \mathcal{H}_∞ -constraints. All present \mathcal{H}_∞ -design approaches and analysis methods are, in one way or another, based on the representation of an \mathcal{H}_∞ -constraint in terms of the eigenvalues of a certain Hamiltonian matrix or matrix constraints such as LMI’s ARE’s or ARI’s. The characterization of an \mathcal{H}_∞ -constraint in terms of a matrix inequality such as an ARI or a LMI is the central tool for the design method in this work. In this section some of the close connections between these different \mathcal{H}_∞ -representations as well as properties of ARI’s and solutions to ARI’s are reviewed.

Lemma 2.2.1 (Frequency Domain)

Consider an asymptotically stable system $G := (A, B, C, D)$, then $\|G\|_\infty < \gamma$ if and only if

$$\gamma^2 I - G^T(-j\omega)G(j\omega) > 0. \quad (2.13)$$

This lemma follows directly from the frequency-domain definition of the \mathcal{H}_∞ -norm and needs no further proof. It simply states that all singular values (as a function of the frequency ω) are smaller than γ for all ω . This lemma is very intuitive and shows, along with the frequency-domain definition of the \mathcal{H}_∞ -norm, that the \mathcal{H}_∞ -norm of a system is a basic system property that does not depend on a particular state-space representation of $G(s)$ or properties such as controllability or observability. The following lemma illustrates the connection between the time-domain definition of the \mathcal{H}_∞ -norm and its frequency-domain counterpart. For further reference the following abbreviations are introduced: $R = (\gamma^2 I - D^T D)$, $S = (\gamma^2 I - D D^T)$. It is easily verified, that R and S satisfy the following relations.

$$\begin{aligned} R^{-1} D^T &= D^T S^{-1} \text{ and} \\ I + D R^{-1} D^T &= \gamma^2 S^{-1}. \end{aligned}$$

Hence in the following equations, R and S are interchangeable using these identities.

Lemma 2.2.2 (M_γ , [11])

Consider a system $G := (A, B, C, D)$ with A asymptotically stable, then $\|G\|_\infty < \gamma$ if and only if M_γ has no purely imaginary eigenvalue where

$$M_\gamma = \begin{pmatrix} A + B R^{-1} D^T C & \gamma B R^{-1} B^T \\ -\gamma C^T S^{-1} C & -[A + B R^{-1} D^T C]^T \end{pmatrix}. \quad (2.14)$$

The relation between lemma 2.2.1 and lemma 2.2.2 is easily established by the fact that M_γ is the system matrix of the state-space representation for $[\gamma^2 I - G^T(-j\omega)G(j\omega)]^{-1}$. It can be shown that $\gamma^2 I - G^T(-j\omega)G(j\omega)$ has a spectral factorization $\gamma^2 I - G^T(-j\omega)G(j\omega) = H^T(-j\omega)H(j\omega)$ if and only if M_γ has no purely imaginary eigenvalue $j\omega$. Thus $\gamma^2 I - G^T(-j\omega)G(j\omega) > 0$ is satisfied if and only if $(j\omega I - M_\gamma)$ is non-singular for all $j\omega$, which in turn is equivalent to $\|G\|_\infty < \gamma$.

Hence the computation of the \mathcal{H}_∞ -norm of a system is reduced to a γ -iteration and a corresponding eigenvalue computation (see [11]). So far, however, no design method is based directly on this property of M_γ . The following lemma illustrates the connection between the time-domain definition of the \mathcal{H}_∞ -norm and its frequency-domain counterpart.

Lemma 2.2.3 (LQ-Cost, ARE, [125])

Consider an asymptotically stable system $G := (A, B, C, D)$ with (A, B) controllable, (C, A) observable and $\gamma > \bar{\sigma}(D)$, then $\|G\|_\infty < \gamma$ if and only if

$$\sup_{w \in L_2} \int_0^\infty [z^T(t)z(t) - \gamma^2 w^T(t)w(t)]dt = \sup_{w \in L_2} J_\infty(w) < \infty. \quad (2.15)$$

If the above supremum is finite then the worst-case disturbance $w_0(t)$ is given by

$$w_0(t) = R^{-1}[D^T C + B^T Y]x(t), \quad (2.16)$$

where $Y = Y^T > 0$ is the unique symmetric positive-definite solution to

$$ARE(Y) = 0, \quad (2.17)$$

$$\begin{aligned} ARE(Y) := & [A + BR^{-1}D^T C]^T Y + Y[A + BR^{-1}D^T C] \\ & + YBR^{-1}B^T Y + \gamma^2 C^T S^{-1}C \end{aligned} \quad (2.18)$$

such that $A + BR^{-1}(D^T C + B^T Y)$ is asymptotically stable and

$$\max_{w \in L_2} J_\infty(w) = x_o^T Y x_o. \quad (2.19)$$

It comes as no surprise that the Hamiltonian matrix associated with this two-point boundary problem is related to M_γ as defined above via a similarity transformation. Hence the Riccati equation (2.17) will have a finite symmetric positive-definite solution Y if and only if the maximization problem has a finite solution. Such a solution exists if and only if M_γ has no $j\omega$ -eigenvalues which in turn is equivalent to the above \mathcal{H}_∞ -bound being satisfied. In general there is a number of matrices \tilde{Y} that satisfy $ARE(\tilde{Y}) = 0$ but only one positive-definite matrix Y that satisfies $ARE(Y) = 0$ such that $A + BR^{-1}(D^T C + B^T Y)$ is asymptotically stable. This matrix Y separates

the spectrum of M_γ into a stable and anti-stable part represented by the eigenvalues of $A + BR^{-1}(D^T C + B^T Y)$ and $-[A + BR^{-1}(D^T C + B^T Y)]^T$ respectively. This lemma forms the basis for all ARE-based methods to \mathcal{H}_∞ -synthesis. In particular the DGKF solution to the \mathcal{H}_∞ -problem is based on this lemma. Equivalent game theoretical approaches (see e.g. [125], [126], [127]) utilize the characterization of an \mathcal{H}_∞ -bound in terms of the above LQ-problem for various control feedback strategies. The obvious relation between the time-domain definition of the \mathcal{H}_∞ -norm and lemma 2.2.3 can be established via a linear fractional optimization problem. The reader is referred to chapter 6 in [20] for more information on this issue. Lemma 2.2.3 is based on variational optimization ideas and invariably invokes the basic assumptions of controllability and observability.

Note at this point that the “if-and-only-if” relationship between $\|G\|_\infty < \gamma$ and the existence of a Y such that $ARE(Y) = 0$ with $A + BR^{-1}(D^T C + B^T Y)$ asymptotically stable is not dependent on its derivation from a variational problem. The considered maximization problem is only a tool to derive this equivalence. Furthermore, if the above controllability condition (or, alternatively observability of (C, A)) is not satisfied, then the corresponding ARE-solution Y can in general be positive semi-definite (see e.g. [17], [43]).

The next lemma provides a very general necessary and sufficient criterion for $\|G\|_\infty < \gamma$ in terms of an ARI.

Lemma 2.2.4 (ARI, [141])

Consider an asymptotically stable system $G := (A, B, C, D)$ and $\gamma > \bar{\sigma}(D)$, then $\|G\|_\infty < \gamma$ if and only if there exists a symmetric positive-definite X_1 such that

$$ARI_1(X_1) < 0, \quad (2.20)$$

$$\begin{aligned} ARI_1(X_1) := & [A + BR^{-1}D^T C]^T X_1 + X_1 [A + BR^{-1}D^T C] \\ & + X_1 B R^{-1} B^T X_1 + \gamma^2 C^T S^{-1} C. \end{aligned} \quad (2.21)$$

This lemma can be derived directly from equation (2.13) and forms the basis for most recent \mathcal{H}_∞ -design methods. An explicit proof can be found in [7] or [141]. Note that the definition of $ARI_1(X_1)$ in (2.21) implies that $ARI_1(X_1)$ is a square matrix and hence the eigenvalues of $ARI_1(X_1)$ are well defined. In the following discussion we will frequently use this fact. In general it is harder to find a solution to a matrix inequality

than a solution to a Riccati equation. There is no “one-step” method available to find solutions for such inequalities. On the other hand, this lemma provides a very general means to determine whether an \mathcal{H}_∞ -bound is satisfied or not. In particular, lemma 2.2.4 does not assume any system properties other than stability. Also, there is not one unique solution to such an inequality. Rather, there is a whole set of possible solutions. It is important to note at this point that this inequality characterization can be viewed as finding a matrix X_1 such that $ARI_1(X_1)$ is negative-definite or in terms of the eigenvalues of the $ARI_1(X_1)$, that is $ARI_1(X_1)$ is asymptotically stable. As the matrix $ARI_1(X_1)$ is a real symmetric matrix, negative definiteness is equivalent to stability of $ARI_1(X_1)$. Hence *any* matrix inequality (or matrix constraint) of the form

$$T[ARI_1(X_1)]T^T < 0, \quad |T| \neq 0, \quad \text{or} \quad (2.22)$$

$$T[ARI_1(X_1)]T^{-1} < 0, \quad TT^T = I, \quad \text{or} \quad (2.23)$$

$$T[ARI_1(X_1)]T^{-1} \text{ asymptotically stable, } |T| \neq 0 \quad (2.24)$$

is equivalent to the ARI criterion in lemma 2.2.4. The constraint $TT^T = I$ in (2.23) is necessary to maintain symmetry of $T[ARI_1(X_1)]T^{-1}$. This fact as well as the Schur complement form of block-structured matrices give rise to some equivalent matrix inequality formulations as presented in the following lemma. They do not form new criteria but rather provide different forms of $ARI_1(X_1)$.

Lemma 2.2.5

Consider an asymptotically stable system $G := (A, B, C, D)$ with $\gamma > \bar{\sigma}(D)$, then the following statements are equivalent:

1.

$$\|G\|_\infty < \gamma. \quad (2.25)$$

2. *There is a system representation $\tilde{G} := (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ such that*

$$ARI_2(T) < 0, \quad (2.26)$$

$$\begin{aligned} ARI_2(T) := & [\tilde{A} + \tilde{B}R^{-1}\tilde{D}^T\tilde{C}]^T + [\tilde{A} + \tilde{B}R^{-1}\tilde{D}^T\tilde{C}] \\ & + \tilde{B}R^{-1}\tilde{B}^T + \gamma^2\tilde{C}^T S^{-1}\tilde{C}, \end{aligned} \quad (2.27)$$

where $\tilde{A} = TAT^{-1}$, $\tilde{B} = TB$, $\tilde{C} = CT^{-1}$ and $\tilde{D} = D$ for some non-singular transformation matrix T . Furthermore, T can be taken to be upper triangular.

3. There is a real symmetric positive semi-definite Z_1 such that

$$LMI_1(Z_1) := \begin{pmatrix} A^T Z_1 + Z_1 A + C^T C & Z_1 B + C^T D \\ B^T Z_1 + D^T C & -(\gamma^2 I - D^T D) \end{pmatrix} < 0. \quad (2.28)$$

4. There is a real symmetric positive semi-definite Z_2 such that

$$LMI_2(Z_2) := \begin{pmatrix} A^T Z_2 + Z_2 A + C^T C & Z_2 B & C^T \\ B^T Z_2 & -\gamma I & D^T \\ C & D & -\gamma I \end{pmatrix} < 0. \quad (2.29)$$

Proof: The equivalence of statements 1 and 2 is a variation of theorem 1 in [112]. Obviously, if (2.25) is true, then there is a symmetric positive-definite solution X_1 to $ARI_1(X_1) < 0$ in (2.20). Now let T be the (non-singular) Cholesky factor of X_1 such that $X_1 = T^T T$, where T is an upper triangular matrix. Now, $ARI_1(X_1) < 0$ if and only if $(T^T)^{-1} ARI_1(X_1) T^{-1} < 0$ for any non-singular matrix T . Multiplying (2.21) with $(T^T)^{-1}$ from the left side and with T^{-1} from the right side yields the desired result, namely equivalence of equations (2.20) and (2.26). Equivalence of statements 1, 3 and 4 is easily shown via repeated application of the Schur complement formula (see Appendix A) for block-structured matrices and is a standard result ([36], [37], [53]). ■

Equation (2.26) shows that a desired \mathcal{H}_∞ -bound can be tested via a search over all nonsingular transformation matrices T , or, alternatively over all possible system realizations of $G(s)$. In general the above inequality characterizations of an \mathcal{H}_∞ -bound will give rise to different numerical schemes to enforce a desired \mathcal{H}_∞ -bound. Hence they are important for the numerical treatment of the corresponding \mathcal{H}_∞ -bound problem in the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design. The inequality formulation in equation (2.28) has the advantage of being linear in Z_1 and γ^2 . The characterization (2.29) is linear in Z_2 and γ . For that reason (2.28) and (2.29) are also referred to as Linear Matrix Inequalities (LMI). However, they are linear in all involved parameters only if the system matrices are assumed to be constant and independent of possible

other variables. When designing an \mathcal{H}_∞ -controller based on the closed-loop \mathcal{H}_∞ -bound characterization in terms of the above LMI's, the matrices A, B, C and D in (2.28) and (2.29) will be functions of the controller. Unfortunately, in the general output-feedback case this dependency of A, B, C and D on the controller parameters will destroy linearity of the above matrix inequality, and ultimately the convexity of the corresponding optimization problem as will be seen later. Further criteria for $\|G\|_\infty < \gamma$ are the Bounded Real Lemma in various forms ([128]) as well as a criterion based on the concept of entropy ([71]). They are not directly relevant to the results in this work and hence are omitted here.

Now let us turn to some properties of possible solutions X_1 for $ARI_1(X_1) < 0$. These properties will prove valuable for the numerical implementation of the \mathcal{H}_∞ - and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design algorithms. To this point let us assume that a particular matrix $X_1^* = (X_1^*)^T > 0$ does indeed satisfy $ARI_1(X_1^*) < 0$. Then there exists a symmetric positive-definite matrix Q such that

$$[A + BR^{-1}D^TC]^T X_1^* + X_1^*[A + BR^{-1}D^TC] + X_1^*BR^{-1}B^T X_1^* + \gamma^2 C^T S^{-1}C + Q = 0, \quad (2.30)$$

where $R = (\gamma^2 I - D^T D)$ and $S = (\gamma^2 I - DD^T)$ as defined previously. Alternatively one can write (2.30) as follows.

$$A^T X_1^* + X_1^* A + [X_1^* B + C^T D]R^{-1}[X_1^* B + C^T D]^T + C^T C + Q = 0. \quad (2.31)$$

If $X_1^* > 0$, $(X_1^*)^{-1}$ is well defined. Now, by use of the above identities on R and S , yet another form of (2.30) can be derived.

$$\begin{aligned} & (X_1^*)^{-1}[A + BD^T S^{-1}C]^T + [A + BD^T S^{-1}C](X_1^*)^{-1} \\ & + \gamma^2 (X_1^*)^{-1}C^T S^{-1}C(X_1^*)^{-1} + BR^{-1}B^T + \tilde{Q} = 0, \end{aligned} \quad (2.32)$$

where $\tilde{Q} = (X_1^*)^{-1}Q(X_1^*)^{-1}$. These equations form the basis for the proof of the following theorem.

Theorem 2.2.1

Given a system $G := (A, B, C, D)$ with $\gamma > \bar{\sigma}(D)$ and $\|G\|_\infty < \gamma$ and a symmetric positive-definite matrix $X_1^* = (X_1^*)^T > 0$ such that $ARI_1(X_1^*) < 0$, then the following statements are true.

1. The system matrix A is asymptotically stable.
2. $A_{aux} = A + BR^{-1}D^TC = A + BD^TS^{-1}C$ is asymptotically stable.
3. $Y \leq X_1^*$, where Y solves $ARE(Y) = 0$ in lemma 2.2.3.
4. $L_o \leq X_1^*$, where L_o solves

$$A^T L_o + L_o A + C^T C = 0. \quad (2.33)$$

5. $\hat{L}_o \leq X_1^*$, where \hat{L}_o solves

$$A_{aux}^T \hat{L}_o + \hat{L}_o A_{aux} + \gamma^2 C^T S^{-1} C = 0. \quad (2.34)$$

6. $\hat{L}_c \leq (X_1^*)^{-1}$, where \hat{L}_c solves

$$\hat{L}_c A_{aux}^T + A_{aux} \hat{L}_c + BR^{-1}B^T = 0. \quad (2.35)$$

Proof: $ARI_1(X_1^*) < 0$ implies the existence of $Q > 0$ and $\tilde{Q} > 0$ in equations (2.30), (2.31) and (2.32) respectively. Now, with

$$[X_1^* B + C^T D] R^{-1} [X_1^* B + C^T D]^T + C^T C + Q > 0,$$

the pair $(A, [X_1^* B + C^T D] R^{-1} [X_1^* B + C^T D]^T + C^T C + Q)$ is observable and hence a standard Lyapunov argument applied to equation (2.31) shows that A must be an asymptotically stable matrix. This proves statement 1. Assertion 2 is shown in the same way by considering equations (2.30) or (2.32) respectively. A proof for statement 3 can be found in [85] and is omitted here. Statement 4 can be proved by subtracting equation (2.33) from equation (2.31) to yield

$$A^T [X_1^* - L_o] + [X_1^* - L_o] A + [X_1^* B + C^T D] R^{-1} [X_1^* B + C^T D]^T + Q = 0.$$

With A asymptotically stable, it follows directly from Lyapunov's theorem that $X_1^* - L_o \geq 0$ and hence statement 4. Statements 5 and 6 can be proved in the same way by use of the ARE's (2.30) and (2.32) and the Lyapunov equations (2.34) and (2.35) respectively. ■

Assume for the moment that the matrices A, B, C and D represent a closed-loop system $G_{cl}(s)$ with a certain controller $C(s)$ in place. Then, if a matrix X_1^* has been found such that $ARI_1(X_1^*) < 0$, two results follow immediately:

1. $\|G_{cl}\|_\infty < \gamma$ by lemma 2.2.4, and
2. $G_{cl} := (A, B, C, D)$ is an asymptotically stable system by theorem 2.2.1.

Hence, an \mathcal{H}_∞ or mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design paradigm based on the ARI-characterization of the corresponding \mathcal{H}_∞ -constraint of the closed-loop system need not enforce stability explicitly. Closed-loop stability will naturally follow once a controller $C(s)$ and a matrix X_1^* have been found such that $ARI_1(X_1^*) < 0$. The other results in theorem 2.2.1 will be valuable for various aspects in the numerical formulation of the proposed algorithm for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design. Note however, that in contrast to lemma 2.2.3 stability of $A + BR^{-1}(D^T C + B^T X_1^*)$ cannot be concluded (and is not necessary any more) from X_1^* satisfying $ARI_1(X_1^*) < 0$. This is due to the fact that theorem 2.2.1 is not based on an optimization problem. The properties of X_1^* in theorem 2.2.1 are based on the frequency-domain inequality (2.13), the equivalent ARI representation in (2.30) and the Lyapunov theorem. No optimal control concepts as in lemma 2.2.3 have been utilized.

To further illustrate the properties stated in theorem 2.2.1, an example plant is considered. The system matrices are as follows.

$$A = \begin{pmatrix} -1.004 & -5.438 & 0 & 0.438 \\ -0.004 & -3.438 & 0 & 0.438 \\ -0.008 & -10.876 & -4.000 & 0.876 \\ -0.218 & 7.743 & 0 & -4.894 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -0.004 & -5.438 \\ 1 & -0.004 & -5.438 \\ 1 & -0.008 & -10.876 \\ 0 & -0.218 & 7.743 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -0.004 & -5.438 & 0 & 0.438 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -0.004 & -5.438 \end{pmatrix}.$$

This plant is asymptotically stable and the corresponding \mathcal{H}_∞ -norm of the system $G := (A, B, C, D)$ is $\|G\|_\infty = 5.4378$. Choosing $\gamma = 5.7000 > \|G\|_\infty$, one particular solution $X_1^* = (X_1^*)^T$ satisfying $ARI_1(X_1^*) < 0$ is

$$X_1^* = \begin{pmatrix} 1.113 & -1.047 & 2.422 & 3.575 \\ -1.047 & 7.348 & -1.494 & -1.880 \\ 2.422 & -1.494 & 39.327 & 56.791 \\ 3.575 & -1.880 & 56.791 & 82.928 \end{pmatrix}. \quad (2.36)$$

The eigenvalues of X_1^* are 0.2869, 0.8215, 7.4424 and 122.1650 and hence $X_1^* > 0$. The corresponding eigenvalues of $ARI_1(X_1^*)$ are -0.1373 , -1.0137 , -1.6807 and -781.1729 , implying that the ARI-inequality constraint $ARI_1(X_1^*) < 0$ is satisfied. It can furthermore be verified that all conditions in theorem 2.2.1 are satisfied as well. However, the eigenvalues of $A + BR^{-1}(D^T C + B^T X_1^*)$ are $0.0417 \pm 1.2607j$ and $0.0217 \pm 0.0732j$, respectively. Hence the matrix $A + BR^{-1}(D^T C + B^T X_1^*)$ is completely antistable. That is, all eigenvalues of $A + BR^{-1}(D^T C + B^T X_1^*)$ are unstable. This fact represents a departure from the ARE-characterization as in lemma 2.2.3, where the solution Y of $ARE(Y) = 0$ needs to satisfy the additional constraint that $A + BR^{-1}(D^T C + B^T Y)$ be asymptotically stable. This is not necessary for the ARI-characterization, as exemplified above.

Let us expand further on this property. Assume an asymptotically stable system $G := (A, B, C, D)$ with $\|G\|_\infty < \gamma$. Then, by lemma 2.2.3 there is a symmetric positive semi-definite matrix Y_o (or positive definite Y_o , depending on the observability and controllability of the pairs (A, B) and (C, A) respectively) satisfying

$$[A + BR^{-1}D^T C]^T Y_o + Y_o[A + BR^{-1}D^T C] + Y_o BR^{-1}B^T Y_o + \gamma^2 C^T S^{-1}C = 0 \quad (2.37)$$

such that $A + BR^{-1}(D^T C + B^T Y_o)$ is asymptotically stable. Now consider possible solutions to the ARE

$$\begin{aligned} & [A + BR^{-1}D^T C]^T (Y_o + dY(\varepsilon)) + (Y_o + dY(\varepsilon))[A + BR^{-1}D^T C] \\ & (Y_o + dY(\varepsilon))BR^{-1}B^T(Y_o + dY(\varepsilon)) + \gamma^2 C^T S^{-1}C + \varepsilon I = 0 \end{aligned} \quad (2.38)$$

for some small positive ε . From (2.37) and (2.38) it follows directly that $dY(\varepsilon)$ satisfies

$$[A + BR^{-1}(D^T C + B^T Y_o)]^T dY(\varepsilon) + dY(\varepsilon)[A + BR^{-1}(D^T C + B^T Y_o)] + \varepsilon I = 0 \quad (2.39)$$

where the quadratic terms in $dY(\varepsilon)$ are neglected. From this Lyapunov equation one can conclude that, with $A + BR^{-1}(D^T C + B^T Y_o)$ asymptotically stable, $dY(\varepsilon) = [dY(\varepsilon)]^T > 0$ is continuous in ε . Thus the eigenvalues of $A + BR^{-1}[D^T C + B^T(Y_o + dY(\varepsilon))]$ are continuous in ε as well. Hence one can choose ε such that $A + BR^{-1}[D^T C + B^T(Y_o + dY(\varepsilon))]$ remains stable. However, $dY(\varepsilon) > 0$ implies $(Y_o + dY(\varepsilon)) > 0$. Hence there exists a symmetric $X_1^o = (Y_o + dY(\varepsilon)) > 0$ such that $ARI_1(X_1^o) < 0$ and $A + BR^{-1}(D^T C + B^T X_1^o)$ is asymptotically stable. Although the above example has shown that in general there may be matrices X_1^* that satisfy the inequality $ARI_1(X_1^*) < 0$ without this additional stability requirement, the above derivation shows that, whenever the \mathcal{H}_∞ -bound is satisfied, there will also exist a symmetric, positive-definite solution to the ARI that also satisfies the additional stability constraint as imposed for the ARE solution. These observations are summarized in the following corollary.

Corollary 2.2.1

Assume an asymptotically stable system $G := (A, B, C, D)$ with $\|G\|_\infty < \gamma$. Then the two following statements are true.

1. *There exists a $X_1^* = (X_1^*)^T > 0$ such that $ARI_1(X_1^*) < 0$.*
2. *There exists a $X_1^o = (X_1^o)^T > 0$ such that $ARI_1(X_1^o) < 0$ and $A + BR^{-1}(D^T C + B^T X_1^o)$ is asymptotically stable.*

So far only strict \mathcal{H}_∞ -bounds have been considered in this discussion, i.e., only the case $\|G\|_\infty < \gamma$. As a final point in this section let us consider the case where $\gamma = \|G\|_\infty$ and its implications for possible solutions to the above ARI's. In the following, assume that A is asymptotically stable and that the necessary condition $\gamma > \bar{\sigma}(D)$ for $\|G\|_\infty \leq \gamma$ is satisfied. In this case the strict inequality (2.13) has to be replaced by $\gamma^2 I - G^T(-j\omega)G(j\omega) \geq 0$. Under these circumstances the proof

for lemma 2.2.4 in [7] is still applicable and it immediately follows that $ARI_1(X_1^*)$ will be negative semi-definite for some X_1^* . However, many of the above conclusions about positive definiteness of X_1^* or stability of A as in theorem 2.2.1 may no longer be true if $ARI_1(X_1^*) \leq 0$. To illustrate these problems, consider equations (2.30) and (2.31), but now for a Q that is positive semi-definite. If $X_1^* > 0$ and $ARI_1(X_1^*) \leq 0$ then A is asymptotically stable. However, X_1^* need not be positive definite in this case. Depending on various observability/controllability conditions, X_1^* may be positive semi-definite or, by similar considerations on (2.32), infinite in some modes. In general, at $\gamma = \|G\|_\infty$ most of the *nice* properties of ARE's and ARI's break down and many important implications become inconclusive. In a numerical implementation this case can be circumvented by forming an “ ε -perturbed” ARI of the form $ARI_1(X_1) + \varepsilon I \leq 0$ which effectively enforces the strict inequality $\|G\|_\infty < \gamma$.

2.3 Uncertain Systems: Stability and Performance

At this point let us consider the problem of robustly stabilizing an uncertain system before adding performance considerations to this scheme. This problem is most generally represented in Figure 2.1.

$\Sigma_{per,op}(s)$ is the open-loop perturbed or uncertain plant model for which a linear stabilizing controller $C(s)$ has to be designed. Uncertainties in the plant description may arise from a variety of sources. These can be neglected dynamics, parametric uncertainties such as component tolerances, model parameter uncertainties or variables that change over the course of operation as well as neglected nonlinearities. Except for some static nonlinearities such as saturated actuators, general nonlinearities cannot be handled with the current \mathcal{H}_∞ -theory. A comprehensive description of these issues can be found in [13]. Mathematically these uncertainties may be represented either in the state-space form as uncertain entries in the respective system matrices, or in the frequency-domain as input and output multiplicative system perturbations $\Delta_i(s)$ and $\Delta_o(s)$ respectively, or as additive system uncertainties $\Delta_a(s)$ as follows.

$$\Sigma_{per,op}(s) = \Sigma_{nom,op}(s)[I + \Delta_i(s)] \quad (2.40)$$

$$\Sigma_{per,op}(s) = [I + \Delta_o(s)]\Sigma_{nom,op}(s) \quad (2.41)$$

$$\Sigma_{per,op}(s) = \Sigma_{nom,op}(s) + \Delta_a(s). \quad (2.42)$$

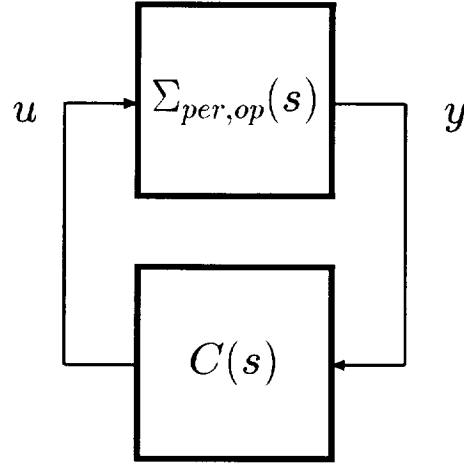


Figure 2.1: Stabilization of an uncertain model.

$\Sigma_{nom,op}(s)$ represents the nominal linear open-loop model. Without any performance objectives, the goal at this point is to find a controller that stabilizes the plant $\Sigma_{per,op}(s)$ for all permissible perturbations $\Delta_i(s)$, $\Delta_0(s)$ or $\Delta_a(s)$. This amounts to stability robustness with respect to uncertainties in the input or output path or to additive uncertainties. \mathcal{H}_∞ -theory requires the uncertainty to be represented in a form that is termed “perturbation feedback form” or “ \mathcal{H}_∞ -standard form”. This representation assumes that all uncertainties are lumped into one uncertainty block $\Delta_s(s)$ that is connected to the nominal plant in a feedback loop as shown in Figure 2.2.

This representation is very general and forms the basis for the application of the Small Gain Theorem to the analysis and synthesis problem in an \mathcal{H}_∞ -setting as well as for the definition of internal stability according to Desoer and Chen and Nyquist-like stability criteria. General frequency-domain uncertainties such as $\Delta_i(s)$, $\Delta_0(s)$ and $\Delta_a(s)$ are easily converted to this form. If one starts with a state-space description of the uncertain open-loop system with parametric uncertainties, this conversion will not always be possible. One way to design robustly stabilizing controllers for this case can be found in [79] and references therein. However, posing the robust stabilization problem in the \mathcal{H}_∞ -standard form has the advantage that it is extendable to necessary

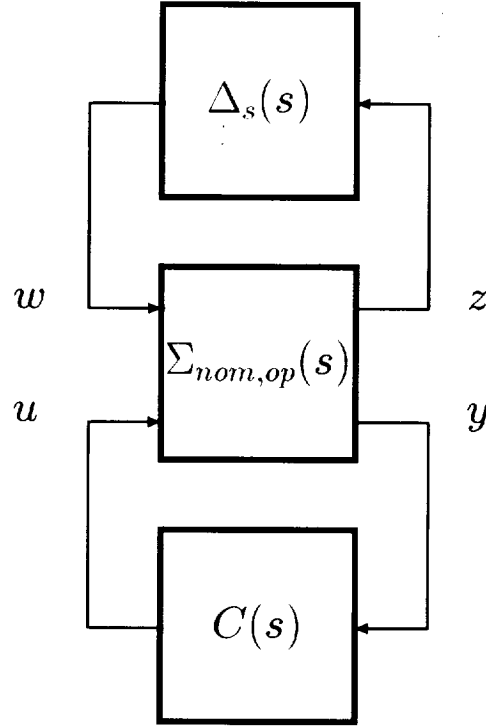


Figure 2.2: Uncertainty representation in \mathcal{H}_∞ -standard form.

and sufficient conditions for robust stability, namely μ -theory. If the state-space matrices of the open-loop system are linear in the uncertain parameters, such a feedback configuration is always possible. This is not necessarily true for the case of multiple uncertainties. Engineering practice has shown that this type of uncertainty description is applicable to a wide range of problems, however. In the following it is assumed that a system representation of the uncertain plant in the \mathcal{H}_∞ -standard form exists. \mathcal{H}_∞ -theory in its present form requires that $\Delta_s(s)$ be stable. A first attempt to include unstable uncertainties $\Delta_s(s)$ into a singular-value based robust stability framework can be found in [54], but the presented theory has to be considered incomplete at this point. Furthermore, the \mathcal{H}_∞ -methodology assumes no internal knowledge of the uncertainty block $\Delta_s(s)$. Structured $\Delta_s(s)$ cannot be incorporated into the pure \mathcal{H}_∞ -design philosophy. An extension of \mathcal{H}_∞ -theory, namely μ -analysis

and design, has to be applied to this type of problem. This subject, however, is not part of the considered research objectives. Henceforth it is explicitly assumed that $\Delta_s(s)$ is a stable transfer function for which no internal structural knowledge is assumed. Let us assume that a controller has been designed and connected to the open-loop system $\Delta_s(s)$ to form the closed-loop system $\Sigma_{nom,cl}(s)$. The question arises, whether the overall system, including the uncertainty block $\Delta_s(s)$, is stable. This question is most elegantly answered by the Small Gain Theorem.

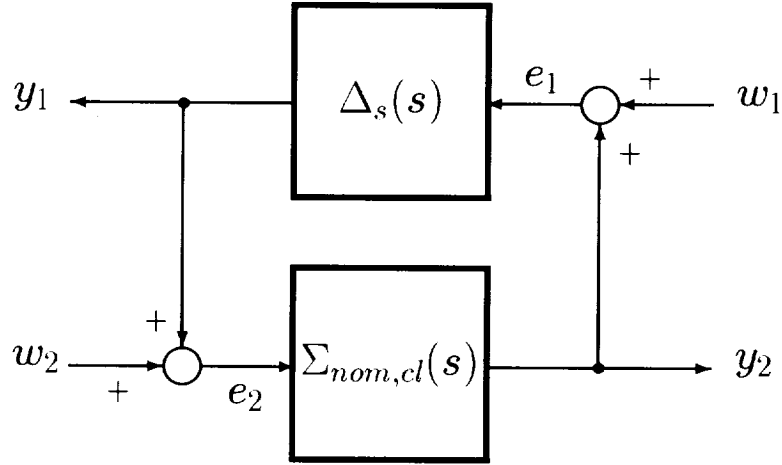


Figure 2.3: Stability and small gain.

Consider an interconnection of these systems as in Figure 2.3 with some auxiliary inputs $w_1(s)$ and $w_2(s)$. This system is internally stable if and only if the four transfer functions from $w_i(s)$ to $e_j(s)$, $i, j = 1, 2$ are asymptotically stable. For this situation the Small Gain Theorem states the following. Assuming that $\Delta_s(s)$ is an asymptotically stable system with a \mathcal{H}_∞ -norm bound $\frac{1}{\gamma}$, i.e. $\|\Delta_s(s)\|_\infty < \frac{1}{\gamma}$ and $\Sigma_{nom,cl}(s)$ is asymptotically stable with \mathcal{H}_∞ -norm bound γ , then the closed-loop system in Figure 2.3 is stable. The proof is most easily performed utilizing the submultiplicativity property of the \mathcal{H}_∞ -norm. It can be verified that

$$\|e_2\|_2 \leq \frac{1}{1 - \|\Delta_s(s)\Sigma_{nom,cl}(s)\|_\infty} \|w_2\|_2. \quad (2.43)$$

Hence from the norm-bound assumptions on the individual transfer functions and $\|\Delta_s \Sigma_{nom,cl}\|_\infty \leq \|\Delta_s\|_\infty \|\Sigma_{nom,cl}\|_\infty < 1$ it follows directly that the closed-loop gain is bounded and hence the transfer function from $w_2(s)$ to $e_2(s)$ is stable. A similar argument establishes stability for the other transfer functions. The synthesis problem can now be stated as follows. Assuming a set of uncertainties that are lumped into an asymptotically stable, \mathcal{H}_∞ -norm bounded system $\Delta_s(s)$ with $\|\Delta_s(s)\|_\infty < \frac{1}{\gamma}$, find a controller $C(s)$ that stabilizes the nominal plant $\Sigma_{nom,op}(s)$ and, in addition satisfies $\|\Sigma_{nom,cl}\|_\infty < \gamma$. This is a nice characterization of robust stability in terms of the \mathcal{H}_∞ -norm. Although there are other criteria to determine whether or not a system is robustly stable, most of these methods have to be considered analysis tools rather than design tools at present.

So far no exogenous input signals have been included into the system description. As mentioned above, external signals may come from a variety of sources. Some signals may be disturbances (deterministic and stochastic) as well as commanded inputs or tracking signals. Fictitious stochastic signals have a long history in the LQG methodology and have proven to be a good means to model sensor and process noises. Although the exact distribution of stochastic disturbances is rarely known precisely, experience and in-depth analysis of the plant environment will in many practical engineering applications permit a close approximation of the noise interference in terms of stochastic signals with known distribution. With appropriate filters these signals can usually be generated from white-noise signals with unit spectral density. These shaping filters are easily incorporated into the open-loop plant model and hence we may consider white-noise signals as the only type of stochastic disturbances entering the plant. In Figure 2.4 $w_2(s)$ collects all stochastic inputs to the plant. That is, $w_2(s)$ includes sensor and process noises as well as other stochastic disturbances and is assumed to contain only white-noise signals with unit spectral density.

Deterministic inputs and other deterministic L_2 -bounded disturbances are represented by the vector $w_{p,\infty}$ in Figure 2.4. As before, it is assumed that all the system uncertainties are lumped into the transfer function $\Delta_s(s)$. From previous considerations it is clear that robust stability can be defined in terms of a \mathcal{H}_∞ -constraint on the transfer function from $w_{s,\infty}(s)$ to $z_{s,\infty}(s)$. Possible weighting functions on this transfer function are assumed to be incorporated into the open-loop model.

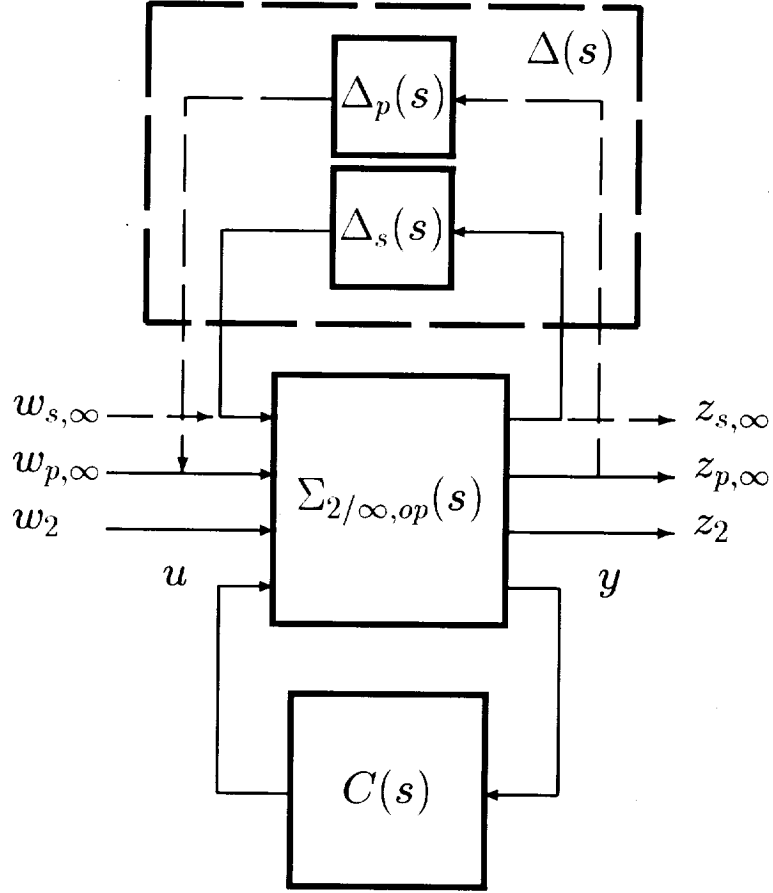


Figure 2.4: Performance with robust stability.

Now let us turn to possible performance objectives in the overall design concept. For this purpose let us define two sets of criterion output vectors $z_{p,\infty}(s)$ and $z_2(s)$. For future reference let $T_2(s)$, $T_{s,\infty}(s)$, $T_{p,\infty}(s)$ and $T_\infty(s)$ denote the transfer functions from $w_2(s)$ to $z_2(s)$, from $w_{s,\infty}(s)$ to $z_{s,\infty}(s)$, from $w_{p,\infty}(s)$ to $z_{p,\infty}(s)$ and from $w_\infty(s) = (w_{s,\infty}^T(s), w_{p,\infty}^T(s))^T$ to $z_\infty(s) = (z_{s,\infty}^T(s), z_{p,\infty}^T(s))^T$ respectively.

In general there is a large number of possible performance specifications that one may want to impose on one or more of the above transfer functions. “Performance” in this context corresponds to any additional requirements on the closed-loop system other than robust stability. Rise and settling times, desirable closed-loop pole loca-

tions, controller gain limitations and so forth are only a few examples. The reader is referred to [13] and references therein for a comprehensive treatment of such performance objectives in an overall design philosophy. Performance in this research refers to \mathcal{H}_2 -performance, although \mathcal{H}_∞ -type performance can also be incorporated.

\mathcal{H}_2 -objectives have a long history and have proven to be a good tool for practical control design tasks. By varying the corresponding weighting matrices Q and R in a corresponding LQ-cost function, many performance objectives can be addressed implicitly with this type of performance criterion. Assume that $w_{p,\infty}(s)$ and $z_{p,\infty}(s)$ are zero and concentrate on the transfer function $T_2(s)$. In this research performance is then defined in terms of the \mathcal{H}_2 -norm of $T_2(s)$. The elements of $z_2(s)$ can correspond to a desired quadratic cost function as in the LQ-framework. More generally, we intend to minimize the effects of the stochastic disturbance signals $w_2(s)$ onto the criterion vector $z_2(s)$. Robust and nominal performance are easily illustrated with the configuration in Figure 2.4. If we can guarantee that the controller minimizes $\|T_2\|_2$ for all permissible $\Delta_s(s)$, then robust \mathcal{H}_2 -performance has been achieved. As mentioned earlier, this problem is still unsolved. Alternatively, one can define the problem of \mathcal{H}_2 -performance for the nominal plant. That is, minimize $\|T_2\|_2$ for $\Delta_s(s) = 0$, subject to robust stability in terms of an \mathcal{H}_∞ -constraint on $T_{s,\infty}(s)$. For the stability problem $w_2(s)$ does not have to be considered and can be assumed to be zero. This problem has the advantage that the corresponding \mathcal{H}_2 -problem and the \mathcal{H}_∞ -problem can be treated separately. This problem has been solved in the most general setting with the least number of system assumptions in this research. In this formulation the “cross transfer functions” from $w_2(s)$ to $z_{s,\infty}(s)$ and from $w_{s,\infty}(s)$ to $z_2(s)$ respectively are neglected. This fact will be discussed in section 3.5 where the results of this work are interpreted in terms of an upper bound for robust \mathcal{H}_2 -performance (see [117]).

Now let us turn to a possible performance objective for the transfer function $T_{p,\infty}(s)$. For deterministic signals $w_{p,\infty}(s)$ the \mathcal{H}_2 -norm has no performance interpretation. For this pair of disturbance/criterion vectors the \mathcal{H}_∞ -norm offers a possible framework to define “performance”. That is, performance may be identified as the worst-case gain of $T_{p,\infty}(s)$ over all frequency and hence in terms of $\|T_{p,\infty}\|_\infty$. Minimizing this norm (or bounding it from above) will reduce the worst-case effect of $w_{p,\infty}(s)$ onto $z_{p,\infty}(s)$ according to equation (2.43). This type of performance is easily transformed into a stability robustness problem by introducing a fictitious uncertainty

block $\Delta_p(s)$ as depicted in Figure 2.4. The overall “stability” problem can be solved via a single \mathcal{H}_∞ -design defined on the transfer function $T_\infty(s)$ corresponding to an overall uncertainty block $\Delta(s)$. This approach results in “robust \mathcal{H}_∞ -performance” as the \mathcal{H}_∞ -bounds are guaranteed for all uncertainties in $\Delta_p(s)$ and $\Delta_s(s)$.

This property is utilized in μ -synthesis to design controllers that provide robust stability and robust performance. However, as mentioned before, this type of performance must not be confused with \mathcal{H}_2 -performance. \mathcal{H}_2 -objectives are not included in μ -synthesis as this type of performance cannot be transformed into a singular-value based stability problem. As in the case of \mathcal{H}_2 , one can define nominal \mathcal{H}_∞ -performance by solving two \mathcal{H}_∞ -problems with the same controller, one for robust stability (with $\Delta_s(s)$ representing model uncertainties), and one for the \mathcal{H}_∞ -performance defined on $T_{p,\infty}(s)$ with $\Delta_p(s)$ as the associated uncertainty block. This formulation requires the capability of solving multiple \mathcal{H}_∞ -constraints, a problem that is still hard to solve in a general formulation.

Let us assume that a possible \mathcal{H}_∞ -performance criterion is defined via a fictitious uncertainty block $\Delta_p(s)$ such that the overall uncertainty $\Delta(s)$ is given by $\Delta(s) = \text{diag} \{ \Delta_s(s), \Delta_p(s) \}$. Referring to Figure 2.5 we are then in a position to state the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -problem (with nominal \mathcal{H}_2 -performance and \mathcal{H}_∞ -robust stability) for the single plant case as follows.

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design strategy:

“Find a controller that minimizes the nominal \mathcal{H}_2 -norm of $T_2(s)$ and robustly stabilizes the closed-loop plant for all uncertainties $\Delta(s)$ subject to $\|\Delta\|_\infty < \frac{1}{\gamma}$ for some prespecified stability radius γ .”

This formulation for the single plant case is easily extended to the multi-plant case as depicted in Figure 2.6, which treats n_p open-loop plants $\Sigma_{2/\infty,op}^i(s)$ for $i = 1, 2, \dots, n_p$ simultaneously. These plants are used to represent different operating points and hence multiple plant conditions or the same plants with multiple \mathcal{H}_2 and/or \mathcal{H}_∞ -objectives. The \mathcal{H}_2 -performance measure is a weighted sum of the individual transfer functions $T_2^i(s)$ from $w_2^i(s)$ to $z_2^i(s)$. Robust stability is defined in terms of n_p \mathcal{H}_∞ -constraints defined on the transfer functions $T_\infty^i(s)$ from $w_\infty^i(s)$ to $z_\infty^i(s)$. This formulation is a natural extension of the above concept for the single plant case and

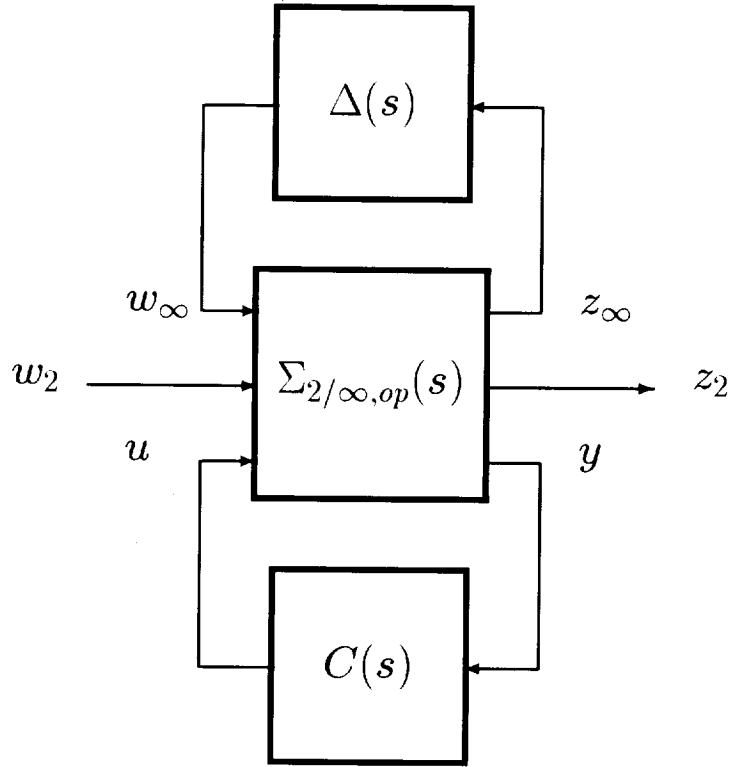


Figure 2.5: The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -synthesis problem – the single-plant case.

represents a general framework for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design problem. It allows the incorporation of multiple \mathcal{H}_∞ -constraints for one or multiple plant models as well as \mathcal{H}_2 -performance for a range of operating conditions of the plant. In the next chapter these objectives will be formulated in a more mathematical fashion along with the corresponding state-space representation and system assumptions.

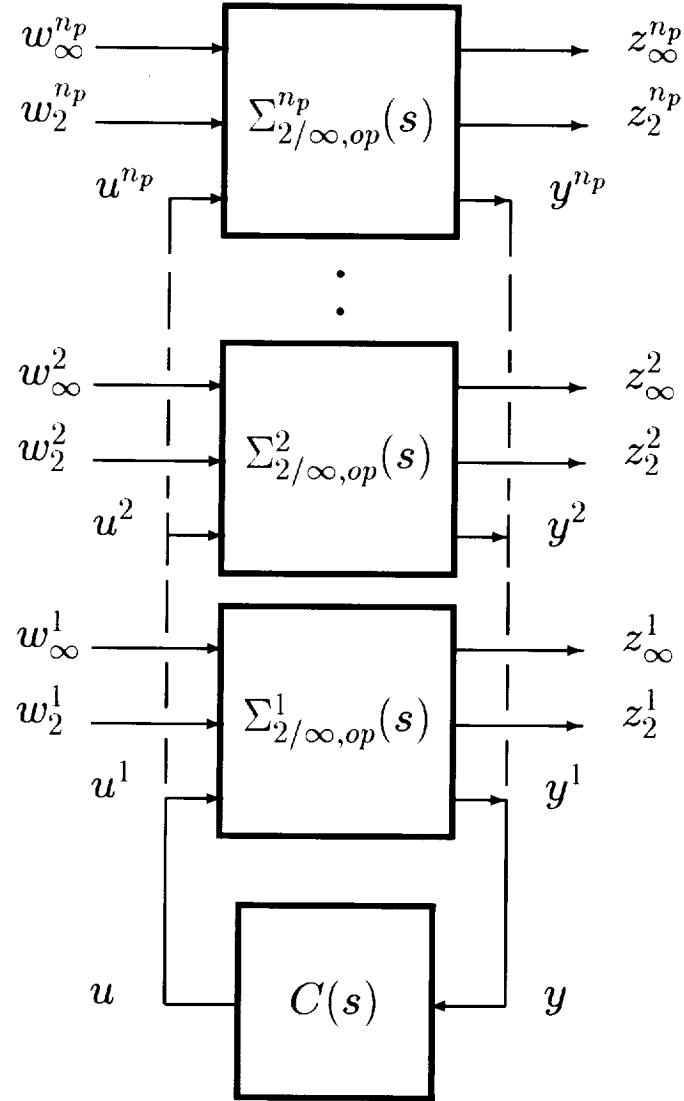


Figure 2.6: The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -synthesis problem – the multi-plant case.

Chapter 3

PROBLEM FORMULATION

3.1 State-Space Description of the Considered Systems

According to the multi-plant mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -formulation defined in chapter 2, we consider n_p linear time-invariant nominal open-loop plants $\Sigma_{2/\infty,op}^i$ with \mathcal{H}_∞ -standard representations for the plant uncertainties. Without loss of generality we assume that each of the n_p individual systems has the following realization.

$$\Sigma_{2/\infty,op}^i : \begin{cases} \dot{x}^i(t) &= A^i x^i(t) + B_1^i w_2^i(t) + B_2^i w_\infty^i(t) + B_3^i u^i(t) \\ z_2^i(t) &= C_1^i x^i(t) + D_{11}^i w_2^i(t) + D_{12}^i w_\infty^i(t) + D_{13}^i u^i(t) \\ z_\infty^i(t) &= C_2^i x^i(t) + D_{21}^i w_2^i(t) + D_{22}^i w_\infty^i(t) + D_{23}^i u^i(t) \\ y^i(t) &= C_3^i x^i(t) + D_{31}^i w_2^i(t) + D_{32}^i w_\infty^i(t) + D_{33}^i u^i(t) \end{cases} \quad (3.1)$$

for $i = 1, 2, \dots, n_p$. $x^i(t)$ represents the i^{th} system state, $u^i(t)$ is the i^{th} control input, $y^i(t)$ is the measurement available to the controller from the i^{th} system, $w_2^i(t)$ and $z_2^i(t)$ are respectively the disturbance input and criterion output for the \mathcal{H}_2 -performance measure on the i^{th} plant, $w_\infty^i(t)$ and $z_\infty^i(t)$ are respectively the disturbance input and criterion output for which the i^{th} \mathcal{H}_∞ -constraint is defined. For a given plant condition, model uncertainties are assumed to be lumped into a stable, norm bounded $\Delta^i(s)$ -block,

$$\|\Delta^i(s)\|_\infty \leq \frac{1}{\gamma^i} \quad (3.2)$$

with the feedback connection

$$w_\infty^i(s) = \Delta^i(s) z_\infty^i(s). \quad (3.3)$$

In general all of the above signals are assumed to have the following dimensions: $x^i(t) \in R^{n_{x^i}}$, $w_2^i(t) \in R^{n_{w_2^i}}$, $w_\infty^i(t) \in R^{n_{w_\infty^i}}$, $z_2^i(t) \in R^{n_{z_2^i}}$, $z_\infty^i(t) \in R^{n_{z_\infty^i}}$, $u^i(t) \in R^{n_u}$ and $y^i \in R^{n_{y^i}}$ for $i = 1, 2, \dots, n_p$. All involved matrices are of compatible dimen-

sions. The controller $C(s)$ is assumed to have the following state-space realization

$$C(s) : \begin{cases} \dot{x}_c(t) &= A_c x_c(t) + B_c y^i(t) \\ u^i(t) &= C_c x_c(t) + D_c y^i(t), \end{cases} \quad (3.4)$$

where $x_c(t) \in R^{n_c}$ and n_c is a prespecified controller dimension. A compact parametric representation of the dynamic controller $C(s)$ is given by

$$C_0 = \begin{pmatrix} D_c & C_c \\ B_c & A_c \end{pmatrix}. \quad (3.5)$$

For static controllers C_0 reduces to $C_0 = D_c$. The system assumptions imposed on the open-loop plants are as follows.

Assumptions:

A1: (A^i, B_3^i) are stabilizable pairs for all $i = 1, 2, \dots, n_p$,

A2: (A^i, C_3^i) are detectable pairs for all $i = 1, 2, \dots, n_p$,

A3: $\dim(u^i) = n_{u^i} = n_u$ and $\dim(y^i) = n_{y^i} = n_y$ for all $i = 1, 2, \dots, n_p$,

A4: $D_{33}^1 = D_{33}^2 = \dots = D_{33}^{n_p} = D_{33}$.

Assumptions A1 and A2 are necessary for the existence of a controller that stabilizes all plant conditions simultaneously. That is, a controller must be able to detect unstable poles through $y^i(s)$ in any of the n_p plants and stabilize these modes via the control $u^i(s)$. The number of controller inputs and outputs must be the same for all $\Sigma_{2/\infty, op}^i$ since we consider only one controller, i.e., one control law for all plant conditions. This necessity is reflected in A3. Assumption A4 is a technical assumption related to a well-posed system for a class of n_p plants controlled by a static controller. When assumption A4 is satisfied and a controller $\tilde{C}(s)$ has been found for the measurement $\tilde{y}^i(s) = y^i(s) - D_{33}u(s)$, then the actual controller for the i^{th} plant condition is

$$u^i(s) = u(s), \quad i = 1, 2, \dots, n_p \quad (3.6)$$

$$u(s) = \tilde{C}(s)[C_3^i x^i(s) + D_{31}^i w_2^i(s) + D_{32}^i w_\infty^i(s)] = \tilde{C}(s)\tilde{y}^i(s), \quad (3.7)$$

$$= \tilde{C}(s)[y^i(s) - D_{33}u(s)]. \quad (3.8)$$

Rearranging equation (3.8) yields ([113])

$$u(s) = [I + \tilde{C}(s)D_{33}]^{-1}\tilde{C}(s)y^i(s) = C(s)y^i(s). \quad (3.9)$$

Thus, if the inverse of $[I + \tilde{C}(s)D_{33}]$ exists, the control problem is well posed and a controller $C(s)$ with the measurement $y^i(s)$ can be found from the control law realized by $\tilde{C}(s)$ using $\tilde{y}^i(s)$ as measurement. Let \tilde{A}_c , \tilde{B}_c , \tilde{C}_c and \tilde{D}_c denote a state-space realization for $\tilde{C}(s)$. Assuming that $(I + \tilde{D}_c D_{33})$ is non-singular, it can be shown that a state-space realization for $C(s)$ is as follows.

$$A_c = \tilde{A}_c - \tilde{B}_c D_{33} (I + \tilde{D}_c D_{33})^{-1} \tilde{C}_c \quad (3.10)$$

$$B_c = \tilde{B}_c (I + D_{33} \tilde{D}_c)^{-1} \quad (3.11)$$

$$C_c = (I + \tilde{D}_c D_{33})^{-1} \tilde{C}_c \quad (3.12)$$

$$D_c = (I + \tilde{D}_c D_{33})^{-1} \tilde{D}_c. \quad (3.13)$$

Thus, under a mild condition and the assumption A4 the controller can be designed by first considering $D_{33} = 0$. The case $D_{33} \neq 0$ can be accounted for after the design for $D_{33} = 0$ has been performed. As the new controller in this case realizes the same control law, properties such as \mathcal{H}_∞ -norms and \mathcal{H}_2 -norms of the closed-loop systems will be preserved under this operation. Hence, in the following we will explicitly assume $D_{33} = 0$. In general it is also possible to remove the direct feedthrough term D_{22} in the above formulation ([113]). This additional term, however, does not increase the complexity of formulae in this presentation. It should also be noted, that the dimensions of the individual open-loop system states $x^i(t)$ are not constrained. In general these dimensions can vary from one plant condition to the other. This fact allows the incorporation of different shaping filters or other dynamics to account for specific requirements of a certain plant condition. Furthermore, unlike the approach taken in [89], no restrictions with respect to system zeros or rank conditions are imposed in this formulation for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design. With the technical assumption A4 the constraints A1–A4 are the minimally necessary assumptions. Hence the proposed formalism provides a versatile and general framework for the considered problem.

Now, given a controller C_0 , the closed-loop plant conditions can be represented as follows.

$$\Sigma_{2/\infty,cl}^i(C_0) : \begin{cases} \dot{x}_{cl}^i(t) &= A_{cl}^i x_{cl}^i(t) &+ B_{cl,2}^i w_2^i(t) &+ B_{cl,\infty}^i w_\infty^i(t) \\ z_2^i(t) &= C_{cl,2}^i x_{cl}^i(t) &&+ D_{cl,\infty,2}^i w_\infty^i(t) \\ z_\infty^i(t) &= C_{cl,\infty}^i x_{cl}^i(t) &+ D_{cl,2,\infty}^i w_2^i(t) &+ D_{cl,\infty}^i w_\infty^i(t). \end{cases} \quad (3.14)$$

Note that the direct feedthrough term $D_{11}^i + D_{13}^i D_c D_{31}^i$ from $w_2^i(t)$ to $z_2^i(t)$ is not shown. The \mathcal{H}_2 -performance measure will be defined on the transfer functions from $w_2^i(t)$ to $z_2^i(t)$ and hence $D_{11}^i + D_{13}^i D_c D_{31}^i = 0$ is a necessary condition for the corresponding \mathcal{H}_2 -norm to be finite. This constraint can be satisfied by directly constraining the structure of the controller $C(s)$ (i.e., design of a strictly proper controller if $D_{11}^i = 0$ for all $i = 1, 2, \dots, n_p$) or has to be added as constraint to the optimization problem to be defined. For the numerical implementation this constraint is assumed to be explicitly satisfied for all plant conditions by a suitable choice of the controller structure depending on the open-loop matrices D_{11}^i , D_{13}^i and D_{31}^i . As discussed earlier, the approach solves the nominal \mathcal{H}_2 -problem subject to \mathcal{H}_∞ -constraints. This implies that we can assume $w_\infty(t) = 0$ for the performance objective. The resulting closed-loop subsystems $\Sigma_{2,cl}^i(C_0)$ from $w_2^i(s)$ to $z_2^i(s)$ with $w_\infty^i = 0$ are as follows.

$$\Sigma_{2,cl}^i(C_0) : \begin{cases} \dot{x}_2^i(t) &= A_{cl,2}^i x_2^i(t) &+ B_{cl,2}^i w_2^i(t) \\ z_2^i(t) &= C_{cl,2}^i x_2^i(t). \end{cases} \quad (3.15)$$

The corresponding subsystems $\Sigma_{\infty,cl}^i(C_0)$ from $w_\infty^i(t)$ to $z_\infty^i(t)$ are defined for $w_2^i(t) = 0$ and are subject to the robust stability criteria in terms of \mathcal{H}_∞ -constraints.

$$\Sigma_{\infty,cl}^i(C_0) : \begin{cases} \dot{x}_\infty^i(t) &= A_{cl,\infty}^i x_\infty^i(t) &+ B_{cl,\infty}^i w_\infty^i(t) \\ z_\infty^i(t) &= C_{cl,\infty}^i x_\infty^i(t) &+ D_{cl,\infty}^i w_\infty^i(t). \end{cases} \quad (3.16)$$

Obviously, in this formulation the (possibly non-zero) direct feedthrough terms $D_{cl,\infty,2}^i$ and $D_{cl,2,\infty}^i$ from $w_\infty^i(s)$ to $z_2^i(s)$ and from $w_2^i(s)$ to $z_\infty^i(s)$, respectively, are not taken into consideration. This fact will be discussed in more detail in section 3.5. In general all the closed-loop state matrices are functions of the controller representation C_0 . This dependency is omitted here to keep the notation to a minimum.

For future reference the closed-loop state-space matrices of $\Sigma_{2,cl}^i(C_0)$ and $\Sigma_{\infty,cl}^i(C_0)$ are rewritten in a form that is convenient for the derivation of the explicit gradient expressions in appendix B.

$$\begin{aligned}
A_{cl,2}^i &= \bar{A}^i + \bar{B}_3^i C_0 \bar{C}_3^i \\
B_{cl,2}^i &= \bar{B}_1^i + \bar{B}_3^i C_0 \bar{D}_{31}^i \\
C_{cl,2}^i &= \bar{C}_1^i + \bar{D}_{13}^i C_0 \bar{C}_3^i \\
A_{cl,\infty}^i &= A_{cl,2}^i = A_{cl}^i \\
B_{cl,\infty}^i &= \bar{B}_2^i + \bar{B}_3^i C_0 \bar{D}_{32}^i \\
C_{cl,\infty}^i &= \bar{C}_2^i + \bar{D}_{23}^i C_0 \bar{C}_3^i \\
D_{cl,\infty}^i &= \bar{D}_{22}^i + \bar{D}_{23}^i C_0 \bar{D}_{32}^i
\end{aligned} \tag{3.17}$$

where

$$\begin{aligned}
\bar{A}^i &= \begin{pmatrix} A^i & 0 \\ 0 & 0 \end{pmatrix}, \\
\bar{B}_1^i &= \begin{pmatrix} B_1^i \\ 0 \end{pmatrix}, \quad \bar{B}_2^i = \begin{pmatrix} B_2^i \\ 0 \end{pmatrix}, \quad \bar{B}_3^i = \begin{pmatrix} B_3^i & 0 \\ 0 & I \end{pmatrix}, \\
\bar{C}_1^i &= \begin{pmatrix} C_1^i & 0 \end{pmatrix}, \quad \bar{C}_2^i = \begin{pmatrix} C_2^i & 0 \end{pmatrix}, \quad \bar{C}_3^i = \begin{pmatrix} C_3^i & 0 \\ 0 & I \end{pmatrix}, \\
\bar{D}_{13}^i &= \begin{pmatrix} D_{13}^i & 0 \end{pmatrix}, \quad \bar{D}_{22}^i = D_{22}^i, \quad \bar{D}_{23}^i = \begin{pmatrix} D_{23}^i & 0 \end{pmatrix}, \\
\bar{D}_{31}^i &= \begin{pmatrix} D_{31}^i \\ 0 \end{pmatrix}, \quad \bar{D}_{32}^i = \begin{pmatrix} D_{32}^i \\ 0 \end{pmatrix}.
\end{aligned} \tag{3.18}$$

Identity matrices in this representation are assumed to be of compatible dimensions such that the matrix operations in (3.17) are well defined. This notation will be maintained throughout the remainder of this report. Note in particular, that this representation shows that all closed-loop matrices are linear in the considered controller parametrization C_0 .

3.2 Problem Definition

Let $T_2^i(C_0, s)$ and $T_\infty^i(C_0, s)$ represent the transfer functions (as a function of the controller parametrization C_0) corresponding to the closed-loop systems $\Sigma_{2,cl}^i(C_0)$ and $\Sigma_{\infty,cl}^i(C_0)$ respectively. Then the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -control problem can be defined as follows.

Definition 3.2.1

Assume n_p open-loop plant conditions as in (3.1) satisfying the assumptions A1 through A4. The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design strategy can be defined as follows: Find a stabilizing controller C_0^* such that the performance criterion $J_2(C_0, t_{f\mathcal{H}_2})$ is minimized where

$$J_2(C_0^*) = \min_{C_0} \lim_{t_{f\mathcal{H}_2} \rightarrow \infty} J_2(C_0, t_{f\mathcal{H}_2}) \quad (3.19)$$

$$J_2(C_0, t_{f\mathcal{H}_2}) = \sum_{i=1}^{n_p} \alpha^i J_2^i(C_0, t_{f\mathcal{H}_2}) \quad (3.20)$$

$$J_2^i(C_0, t_{f\mathcal{H}_2}) = \mathcal{E}[z_2^{iT}(t_{f\mathcal{H}_2}) z_2^i(t_{f\mathcal{H}_2})], \quad (3.21)$$

where

$$\lim_{t_{f\mathcal{H}_2} \rightarrow \infty} J_2^i(C_0, t_{f\mathcal{H}_2}) = \|T_2^i(C_0)\|_2^2, \quad (3.22)$$

subject to the constraints

$$\|T_\infty^i(C_0)\|_\infty < \gamma^i \quad (3.23)$$

for a stabilizing controller C_0 and all $i = 1, 2, \dots, n_p$. The n_p parameters γ^i are chosen by the designer and α^i are n_p weighting factors.

This formulation is the mathematical equivalent to the problem posed in the last chapter. The \mathcal{H}_2 -performance index $J_2(C_0^*)$ is the weighted sum of the individual (nominal) \mathcal{H}_2 -norms for each plant condition while robust stability is imposed via n_p \mathcal{H}_∞ -constraints. This is a constrained optimization problem where the \mathcal{H}_∞ -bound can be expressed in terms of any of the characterizations presented in the last chapter. In the overall optimization the performance cost can be expressed in terms of a finite-time cost or in terms of the system grammians which correlate to $t_{f\mathcal{H}_2} = \infty$ in the above definition.

3.3 The \mathcal{H}_2 -design problem

Depending on whether the initial controller guess is stabilizing or not there are two possible ways of setting up the corresponding \mathcal{H}_2 -problem computationally. If the initial controller guess is stabilizing, the i^{th} performance cost can be computed via the controllability or observability grammians as follows.

$$\|T_2^i(C_0)\|_2^2 = \text{Trace}[(B_{cl,2}^i)^T L_o^i B_{cl,2}^i] \quad (3.24)$$

$$= \text{Trace}[C_{cl,2}^i L_c^i (C_{cl,2}^i)^T] \quad (3.25)$$

where L_o^i and L_c^i solve

$$(A_{cl}^i)^T L_o^i + L_o^i A_{cl}^i + (C_{cl,2}^i)^T C_{cl,2}^i = 0 \quad (3.26)$$

$$A_{cl}^i L_c^i + L_c^i (A_{cl}^i)^T + B_{cl,2}^i (B_{cl,2}^i)^T = 0. \quad (3.27)$$

On the other hand, if C_0 is not stabilizing the i^{th} plant, $J_2^i(C_0, t_{f\mathcal{H}_2})$ can be expressed in terms of a finite-time cost function as

$$J_2^i(C_0, t_{f\mathcal{H}_2}) = \text{Trace}\left[\int_0^{t_{f\mathcal{H}_2}} (B_{cl,2}^i)^T e^{(A_{cl}^i)^T t} (C_{cl,2}^i)^T C_{cl,2}^i e^{A_{cl}^i t} B_{cl,2}^i dt\right] \quad (3.28)$$

$$= \text{Trace}\left[\int_0^{t_{f\mathcal{H}_2}} C_{cl,2}^i e^{A_{cl}^i t} B_{cl,2}^i (B_{cl,2}^i)^T e^{(A_{cl}^i)^T t} (C_{cl,2}^i)^T dt\right]. \quad (3.29)$$

If the closed-loop system is stable, then in the limit as $t_{f\mathcal{H}_2} \rightarrow \infty$, (3.28) and (3.29) are equivalent to (3.24) and (3.25). The infinite-time approach in (3.24) or (3.25) is well known and is used in all current design approaches to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -problem (e.g. [89]). An \mathcal{H}_2 -approach based on a finite-time cost function in (3.28) and (3.29) was introduced in [64], where corresponding explicit gradient expressions were derived. The concept has been applied successfully to a variety of \mathcal{H}_2 -problems. As already mentioned, this approach does not require an initially stabilizing controller. However, if the pure \mathcal{H}_2 -problem is considered, the requirement for closed-loop stability has to be augmented in terms of constraints on the eigenvalues of A_{cl}^i for all plant conditions if certain observability/controllability conditions are violated. In either case, these representations are smooth and hence can be solved using standard gradient-based software.

3.4 The \mathcal{H}_∞ -design problem

3.4.1 Notational Convention for ARI's and LMI's

In chapter 2 a particular notational convention was not so important as no specific control strategy is considered there. Only closed-loop systems with an independent parametrization in terms of (A, B, C, D) are analyzed. When dealing with specific feedback control problems a more specific notation is needed. In the following “LMI” will refer to matrix inequalities that are linear in the sought-after solution X (see lemma 2.2.5) while “ARI” denotes Riccati-type inequalities of the various types presented in chapter 2. In particular, the following forms (and parameter dependencies) for ARI's are used consistently throughout the remainder of this report.

1. $ARI_{C,OF}^i(C_0, X^i, \gamma^i)$:
Algebraic Riccati Inequality for the i^{th} \mathcal{H}_∞ -bound associated with the i^{th} continuous-time (subscript C) plant condition for the output-feedback case (subscript OF) as a function of the controller matrix C_0 , the sought-after solution X^i for the i^{th} ARI, and γ^i .
2. $ARI_{C,SF}(C_0, X, \gamma)$:
Algebraic Riccati Inequality for the \mathcal{H}_∞ -bound associated with the continuous-time (subscript C) plant for the full state-feedback case (subscript SF) as a function of the controller matrix C_0 , the sought-after solution X for the ARI, and γ . The plant index i is dropped in this case (see section 6.2).
3. $ARI_{D,SF}(C_0, X, \gamma)$ (or $LMI_{D,SF}(C_0, X, \gamma)$):
Algebraic Riccati Inequality (Linear Matrix Inequality) representing the \mathcal{H}_∞ -constraint in the discrete-time domain (subscripts D) full state-feedback case (subscript SF) for a single plant (see appendix D).

This notational complexity is necessary to address a wide variety of problems associated with the \mathcal{H}_∞ - and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -problems considered in this work. In particular the dependence on the set γ^i for the general output-feedback case and on γ for the state-feedback case is included to extend the problem formulation to the optimal \mathcal{H}_∞ -problem. With this convention we can now reformulate the \mathcal{H}_∞ -constraints in terms of matrix inequalities.

3.4.2 Reformulation of \mathcal{H}_∞ -Constraints in Terms of ARI's/LMI's

The \mathcal{H}_∞ -problem in definition 3.2.1 is not as readily amenable to gradient-based methods. If the basic frequency-domain definition for the \mathcal{H}_∞ -norm is used to represent the desired \mathcal{H}_∞ -constraints in definition 3.2.1, then the resulting overall optimization problem is not smooth. Alternatively the \mathcal{H}_∞ -constraints may be expressed in terms of n_p ARE's. This approach was taken in [89] and in [112], where a Lagrange multiplier approach has been utilized to append the \mathcal{H}_∞ -constraints in terms of ARE's to the \mathcal{H}_2 -performance cost. This formulation, however, imposes a variety of restrictive system assumptions as brought forth in [24]. In this research the \mathcal{H}_∞ -constraints on the closed-loop systems are replaced by n_p matrix inequalities. Any of the ARI's (or the LMI) in lemmas 2.2.4 or 2.2.5 can be used for this purpose. It has to be kept in mind, that these inequalities will now be functions of the controller C_0 and their respective solutions X^i . As a result of the above discussion the n_p \mathcal{H}_∞ -constraints for the general output-feedback case as posed in definition 3.2.1 can now be reformulated as follows.

Definition 3.4.1

Consider n_p closed-loop plant conditions as in (3.16) satisfying the assumptions A1 through A4. Then the suboptimal \mathcal{H}_∞ -design problem can be posed as follows: Find a stabilizing controller C_0 and a set of n_p matrices X^i such that the following constraints are satisfied,

$$\begin{aligned} 1.) \quad & \text{ARI}_{C,OF}^i(C_0, X^i, \gamma^i) < 0 \\ 2.) \quad & D_{cl,\infty}^{iT} D_{cl,\infty}^i - (\gamma^i)^2 I < 0 \\ 3.) \quad & -X^i < 0 \\ 4.) \quad & X^i = X^{iT} \end{aligned} \tag{3.30}$$

for a given a set of n_p \mathcal{H}_∞ -bounds γ^i ($i = 1, 2, \dots, n_p$) as in definition 3.2.1 where

$$\begin{aligned} \text{ARI}_{C,OF}^i(C_0, X^i, \gamma^i) = & [A_{cl}^i + B_{cl,\infty}^i (R^i)^{-1} (D_{cl,\infty}^i)^T C_{cl,\infty}^i]^T X^i \\ & + X^i [A_{cl}^i + B_{cl,\infty}^i (R^i)^{-1} (D_{cl,\infty}^i)^T C_{cl,\infty}^i] \\ & + X^i B_{cl,\infty}^i (R^i)^{-1} (B_{cl,\infty}^i)^T X^i + (\gamma^i)^2 (C_{cl,\infty}^i)^T (S^i)^{-1} C_{cl,\infty}^i \end{aligned} \tag{3.31}$$

$$R^i = (\gamma^i)^2 I - (D_{cl,\infty}^i)^T D_{cl,\infty}^i \tag{3.32}$$

$$S^i = (\gamma^i)^2 I - D_{cl,\infty}^i (D_{cl,\infty}^i)^T. \tag{3.33}$$

Constraint 2 is a necessary condition for $\|T_\infty^i(C_0, s)\|_\infty < \gamma^i$, conditions 3 and 4 specify the desired solution for the ARI and constraint 1 enforces the matrix inequality constraint itself.

In general, if a controller C_0 can be found that satisfies all these constraints, then $ARI_{C,OF}^i(C_0, X^i, \gamma^i) < 0$ implies $\|T_\infty^i(C_0)\|_\infty < \gamma^i$. Unfortunately nothing can be said about the gap between $\|T_\infty^i(C_0)\|_\infty$ and the specified γ^i . That is, no means are available to determine “how close” the achieved \mathcal{H}_∞ -norm of the i^{th} closed-loop plant is to the specified robust stability bound γ^i . This gap will in general depend on the plant data, the controller and the “distance” between the eigenvalue $\bar{\lambda}[ARI_{C,OF}^i(C_0, X^i, \gamma^i)] < 0$ and the origin. This formulation immediately poses the question of how to enforce matrix inequalities in a gradient-based formulation. A new novel method to reformulate constraints of this kind in terms of a scalar cost function will be introduced and discussed in chapter 4.

3.4.3 State of the Art in \mathcal{H}_∞ -Synthesis

Based on the Youla parametrization (see e.g. [66]) it can be shown that the \mathcal{H}_∞ -design problem is in general infinite dimensional. The class of all suboptimal \mathcal{H}_∞ -controllers can be constructed from an arbitrary stabilizing controller and an additional (infinite-dimensional) \mathcal{H}_∞ -norm-bounded transfer function $Q(s)$. This led to frequency-domain methods solving the so-called One-, Two- and Four-Block problems. The main thrust for the application of \mathcal{H}_∞ -methods to practical control problems were the DGKF equations ([24]). The parametrization of \mathcal{H}_∞ -suboptimal controllers in terms of two Riccati equations provided an elegant and numerically tractable design method for the computation of \mathcal{H}_∞ -suboptimal controllers that are of the same dimension as the open-loop plant. The main drawback of this solution is the set of rather restrictive system assumptions imposed on the open-loop system. These assumptions include restrictions on the system zeros and rank conditions on various system matrices. Due to the rank assumptions this approach is known as the *regular \mathcal{H}_∞ -problem* and the resulting controller based on this approach is the *Central Controller* (as $Q(s)$ – from the Youla parametrization – is assumed to be zero). The rank assumptions in the DGKF approach were removed in the work by Stoorvogel ([113], [115]). There \mathcal{H}_∞ -suboptimal controllers are computed via the solution of two

quadratic matrix inequalities subject to rank constraints on two subsystems of the open-loop plant. Due to the removal of the rank assumptions this approach solves the so-called *singular* \mathcal{H}_∞ -problem. However, this approach still requires assumptions on the open-loop system zeros of the plants under consideration. Until recently this problem had to be circumvented via ε -perturbation techniques ([97], [45]).

A new type of approach was initiated by Sampei et.al. ([98]) where the \mathcal{H}_∞ -problem for output-feedback controllers has been posed in terms of two ARI's. The system assumptions involve a minimal set of necessary constraints on the open-loop system, namely detectability and stabilizability. No further constraints are imposed. This was the first approach that removed the restrictive assumptions on the system zeros. This idea has been developed further and resulted in a (convex) parametrization of all (full-order) \mathcal{H}_∞ -suboptimal controllers. Early exposition of this method may be found in [35], [36] and subsequently in [52]. The corresponding theorem is stated here (with the notation used in this report) for the sake of completeness and as a comparison to the approach taken in the presented formalism.

Theorem 3.4.1 (Theorem 3.1 in [52])

Consider a controller of order n_c and the \mathcal{H}_∞ -problem for the single plant case, i.e., $n_p = 1$. Then the following statements are equivalent.

1. *There exists a controller of order n_c such that $\|T_\infty(C_0, s)\|_\infty < \gamma$.*
2. *There are symmetric positive-definite matrices X and Y such that*

$$L \begin{pmatrix} AX + XA^T + B_2 B_2^T & XC_2^T + B_2 D_{22}^T \\ C_2 X + D_{22} B_2^T & D_{22} D_{22}^T - \gamma^2 I \end{pmatrix} L^T < 0 \quad (3.34)$$

$$M^T \begin{pmatrix} YA + A^T Y + C_2^T C_2 & YB_2 + C_2^T D_{22} \\ B_2^T Y + D_{22}^T C_2 & D_{22}^T D_{22} - \gamma^2 I \end{pmatrix} M < 0 \quad (3.35)$$

$$\begin{pmatrix} X & I \\ I & Y \end{pmatrix} \geq 0 \quad (3.36)$$

$$\text{rank}(I - XY) \leq n_c \quad (3.37)$$

where the columns of L form a basis for $\mathcal{N}[\begin{pmatrix} B_3 \\ D_{23} \end{pmatrix}]$ and the columns of M represent a basis for $\mathcal{N}[\begin{pmatrix} C_3 & D_{32} \end{pmatrix}]$.

A parametrization of the controller $C(s)$ in terms of X and Y is given in theorem 3.2 in [52]. This formulation constitutes the “state of the art” in \mathcal{H}_∞ -synthesis, as all suboptimal \mathcal{H}_∞ -controllers of order $n_c = n_x$ can be derived from the solution of two linear matrix inequalities along with a coupling condition. Unlike in the LQG-philosophy the separation principle is no longer valid. That is, the optimal output estimation and full information control problem (the \mathcal{H}_∞ -equivalent to the full state-feedback problem in LQG-theory, see [43]) cannot be treated separately in \mathcal{H}_∞ -control. The coupling condition (3.36) is a necessary condition for closed-loop stability. The above theorem completely characterizes all full-order \mathcal{H}_∞ -controllers (i.e. $n_c = n_x$) in terms of the three inequalities (3.34), (3.35) and (3.36). It imposes the least number of system restrictions and, in particular, does not make any assumptions on the system zeros.

For fixed-order controllers, however, the additional constraint (3.37) has to be satisfied. Rank constraints are hard problems to solve and are the subject of on-going research (see e.g. [36], [37]). Furthermore, the controller parametrization follows directly from the solutions X and Y of the above inequalities. Hence this approach is not applicable to design problems where the controller is structurally constrained or problems with multiple plants as proposed in this work. Most importantly, the above formulation also leads to the fundamental problem of how to enforce matrix inequalities. This problem will be addressed in the next chapter.

3.5 Robust \mathcal{H}_2 -Performance – Some Recent Results

The problem of robust \mathcal{H}_2 -performance is to a large extent still an unsolved problem, as pointed out in the introduction. However, some promising results for this problem have been derived in [39] and [83]. An ARE-based approach is used to define an upper bound for the \mathcal{H}_2 -cost for all considered uncertainties with a given \mathcal{H}_∞ -norm bound. A different approach to define an upper bound for the robust \mathcal{H}_2 -performance measure has been investigated in [114] and [117]. The approach is intuitive and is briefly reviewed here as it allows an interpretation – and possible extension – of the presented results to the problem of robust \mathcal{H}_2 -performance. For this purpose consider

the n_p closed-loop systems $\Sigma_{2/\infty,cl}^i(C_0)$ in (3.14),

$$\Sigma_{2/\infty,cl}^i(C_0) : \begin{cases} \dot{x}_{cl}^i(t) &= A_{cl}^i x_{cl}^i(t) + B_{cl,2}^i w_2^i(t) + B_{cl,\infty}^i w_\infty^i(t) \\ z_2^i(t) &= C_{cl,2}^i x_{cl}^i(t) + D_{cl,\infty,2}^i w_\infty^i(t) \\ z_\infty^i(t) &= C_{cl,\infty}^i x_{cl}^i(t) + D_{cl,2,\infty}^i w_2^i(t) + D_{cl,\infty}^i w_\infty^i(t) \end{cases}$$

with the uncertainty block $\Delta^i(s)$ connected to this system in a feedback connection such that $w_\infty^i(s) = \Delta^i(s)z_\infty^i(s)$. For the results derived in [117], $\Delta^i(s)$ may be linear, nonlinear, time-varying or time-invariant, providing a generalization of the standard \mathcal{H}_∞ -assumption on the uncertainties. Without any knowledge on the internal structure of $\Delta^i(s)$ the uncertainty is allowed to contain a “direct feedthrough term”. With $D_{cl,2,\infty}^i \neq 0$ this would cause a direct feedthrough term from $w_2^i(s)$ to $z_\infty^i(s)$ and through the uncertainty to $z_2^i(s)$. As a result the \mathcal{H}_2 -norm of the transfer function from $w_2^i(s)$ to $z_2^i(s)$ would be infinite. Thus as in [117] we need to ensure that $D_{cl,2,\infty}^i = 0$. Although having $D_{cl,\infty,2}^i = 0$ would yield the same result, it is assumed here that $D_{cl,2,\infty}^i = 0$ with possible $D_{cl,\infty,2}^i \neq 0$. For this case the relevant transfer functions and their corresponding input/output mappings and state-space realizations are given in terms of the closed-loop matrices. Here a closed-loop system is related to the system configuration with a controller C_0 and not with the uncertainty $\Delta^i(s)$.

$$\begin{aligned} T_\infty^i(C_0, s) &: w_\infty^i(s) \mapsto z_\infty^i(s), & T_\infty^i(C_0) &:= [A_{cl}^i, B_{cl,\infty}^i, C_{cl,\infty}^i, D_{cl,\infty}^i] \\ T_{2,\infty}^i(C_0, s) &: w_2^i(s) \mapsto z_\infty^i(s), & T_{2,\infty}^i(C_0) &:= [A_{cl}^i, B_{cl,2}^i, C_{cl,\infty}^i, 0] \\ T_{\infty,2}^i(C_0, s) &: w_\infty^i(s) \mapsto z_2^i(s), & T_{\infty,2}^i(C_0) &:= [A_{cl}^i, B_{cl,\infty}^i, C_{cl,2}^i, D_{cl,\infty,2}^i] \\ T_2^i(C_0, s) &: w_2^i(s) \mapsto z_2^i(s), & T_2^i(C_0) &:= [A_{cl}^i, B_{cl,2}^i, C_{cl,2}^i, 0]. \end{aligned} \quad (3.38)$$

Obviously $T_2^i(C_0, s)$ and $T_\infty^i(C_0, s)$ are the closed-loop transfer functions for the nominal problem. With the additional cross coupling terms $T_{2,\infty}^i(C_0, s)$ and $T_{\infty,2}^i(C_0, s)$ along with the i^{th} uncertainty $\Delta^i(s)$ it can be shown that the transfer function $T_{2,\Delta}^i(C_0, s)$ from $w_2^i(s)$ to $z_2^i(s)$ with both the controller C_0 and the uncertainty block $\Delta^i(s)$ closed in the i^{th} system has the following form.

$$T_{2,\Delta}^i(C_0, s) = T_2^i(C_0, s) + T_{\infty,2}^i(C_0, s)[I - \Delta^i(s)T_\infty^i(C_0, s)]^{-1}\Delta^i(s)T_{2,\infty}^i(C_0, s). \quad (3.39)$$

By applying appropriate system norm inequalities it can be shown that (lemma 2.3, [117])

$$\|T_{2,\Delta}^i(C_0)\|_2 \leq \|T_2^i(C_0)\|_2 + \frac{\|T_{\infty,2}^i(C_0)\|_\infty \|\Delta^i\|_\infty}{1 - \|\Delta^i\|_\infty \|T_\infty^i(C_0)\|_\infty} \|T_{2,\infty}^i(C_0)\|_2. \quad (3.40)$$

This inequality gives rise to a range of observations and interpretations for the nominal \mathcal{H}_2 -design formulation in this work. For the nominal \mathcal{H}_2 -problem obviously the term $T_{\infty,2}^i(C_0, s)$ is assumed to be zero and hence the second term on the right side of inequality (3.40) is neglected. In face of (3.40) and with the capability to include multiple plant conditions, one can define an “upper bound robust \mathcal{H}_2 -performance problem with robust stability” using the following design objectives

1. Minimize a weighted sum of $\|T_2^i(C_0)\|_2$ and $\|T_{2,\infty}^i(C_0)\|_2$,
2. Minimize the \mathcal{H}_∞ -norms of $T_\infty^i(C_0, s)$ and $T_{\infty,2}^i(C_0, s)$ for given \mathcal{H}_∞ -norm bounds on the uncertainties $\Delta^i(s)$,
3. Guarantee robust stability for all plant conditions in terms of the Small Gain Theorem by $\|\Delta^i\|_\infty \|T_\infty^i(C_0)\|_\infty < 1$,

for all i , $i = 1, 2, \dots, n_p$. It should be noted, that inequality (3.40) does not place any assumptions on the uncertainties $\Delta^i(s)$, not even causality. This may cause the upper bound in (3.40) to be very conservative. At this point there is no theoretical means to compute the gap between the actual worst-case norm $\|T_{2,\Delta}^i(C_0)\|_2$ and the upper bound. The true robust \mathcal{H}_2 -performance problem – not an upper bound – is a problem that has yet to be solved. However, inequality (3.40) has implications for the nominal \mathcal{H}_2 -problem. In particular, after a design has been performed for the nominal problem, it is always possible to compute an upper bound for the robust \mathcal{H}_2 -performance given by the right-hand side of inequality (3.40).

Although these ideas are not pursued further in this work, the stated results allow the designer to easily compute a guaranteed \mathcal{H}_2 -performance cost for all possible uncertainties and hence an upper bound for the robust \mathcal{H}_2 -performance cost at the i^{th} plant condition, regardless of which design method is used to derive the controller.

Chapter 4

SYMMETRIC MATRIX-INEQUALITIES, THEIR ROLE IN CONTROL SYSTEMS THEORY AND A NEW COST FUNCTION FOR THEIR ENFORCEMENT

4.1 *Matrix Inequalities*

A wide variety of control problems can be reduced to matrix inequalities. In the past such criteria have largely been neglected as they do not allow the computation of analytical one-step solutions. The representation of \mathcal{H}_∞ -constraints is one of many problems that can be casted as a matrix inequality constraint. Stability, for example, can also be characterized in terms of parameter dependent Lyapunov inequalities. Present approaches to μ -synthesis convert the general min-max problem to a sequence of optimization problems involving the minimization of the \mathcal{H}_∞ -norm of a scaled constant matrix. This formulation in turn can be expressed in terms of matrix inequalities. Present \mathcal{H}_∞ -synthesis methods often yield unstable controllers which are undesirable in practice. The requirement for a stable controller, however, can be easily translated into a Lyapunov-type inequality constraint. Other applications of matrix inequalities are optimization problems involving maximum generalized eigenvalues, minimum dissipation constraints, Hankel norm constraints, inverse optimal control problems and many more. A forthcoming book by Boyd et. al. contains a very complete list of such problems and their conversion to matrix inequalities (see [15]). Presently, however, there are no reliable gradient-based methods available to find solutions for such inequalities.

In the following let us consider a general problem as follows. Let $\mathcal{Q}(v)$ be a real symmetric matrix-valued function of a set of independent real variables v . That is, the expression $\mathcal{Q}(v)$ defines a mapping of the variables v to the general set of symmetric matrices as follows.

$$\mathcal{Q}(v) : v \mapsto \mathcal{Q}(v), \quad v \in R^p, \quad \mathcal{Q}(v) = [\mathcal{Q}(v)]^T \in R^{q \times q}. \quad (4.1)$$

For this kind of matrix-valued function we consider the following inequality constraint

$$\mathcal{Q}(v) < 0. \quad (4.2)$$

This formulation is general and includes all the above control problems with proper definitions of $\mathcal{Q}(v)$.

4.2 Present Solution Methods

Present approaches to enforce matrix inequalities include non-differentiable methods such as Kelley's cutting plane or ellipsoid methods. These methods are based on subgradients and cannot be integrated into a gradient-based formulation. Other approaches such as homotopy methods have been successfully applied to solve these problems. The only differentiable scalar function defined for the computation of solutions to inequalities is an "interior point" method and can be found in [15], [14]. There the following barrier function has been defined.

$$\phi(v) = \begin{cases} \log \det[-\mathcal{Q}(v)]^{-1} & v \in \mathcal{V} \\ \infty & v \notin \mathcal{V}, \end{cases} \quad (4.3)$$

where \mathcal{V} is the set of feasible solutions such that $v \in \mathcal{V} \Rightarrow \mathcal{Q}(v) < 0$. This type of barrier function has been successfully applied to scalar inequality constraints (see e.g. [63]). For matrix inequalities, however, a closer look reveals that $\phi(v)$ is a barrier function only by definition, not by virtue. That is, one can easily find problems where $\mathcal{Q}(v)$ is sign-indefinite or even positive definite and $\log \det[-\mathcal{Q}(v)]^{-1}$ still remains finite. This will happen whenever the number of positive eigenvalues of $\mathcal{Q}(v)$ is even. In general, when moving from the scalar case to the matrix case one has to take into account the multi-dimensional nature of the problem at hand. Also, because $\phi(v)$ is an interior point method, the initial guess for this method must a-priori satisfy the constraint. Hence this function is applicable in a gradient-based optimization only when a solution for the desired constraint is known a-priori. Presently such a "guess" is generated using the non-gradient based methods listed above. In the following section a new time-domain cost function is defined that removes the problems associated with $\phi(v)$.

4.3 A Scalar Differentiable Cost Function to Enforce Matrix Inequalities

In this section we consider the problem posed in equations (4.1) and (4.2). For this feasibility problem we define the following time-domain penalty-function $f(v, t_f)$. It is a function of the independent variables v and an auxiliary positive scalar parameter t_f . Namely,

$$f(v, t_f) = \text{Trace}\{e^{[\mathcal{Q}(v)]t_f}\}. \quad (4.4)$$

This cost function is an extension of scalar penalty functions to the matrix case and possesses many attractive properties as elaborated in the following section.

Due to the fact that $\mathcal{Q}(v)$ is symmetric, negative definiteness of $\mathcal{Q}(v)$ is equivalent to $\mathcal{Q}(v)$ being stable. The matrix $e^{[\mathcal{Q}(v)]t_f}$ can be interpreted as the transition matrix of a system $\dot{e}(t) = \mathcal{Q}(v)e(t)$ and hence the inequality constraint $\mathcal{Q}(v) < 0$ can be interpreted as a stabilization problem. This justifies the classification of $f(v, t_f)$ as a time-domain function. It is easily verified that the defined cost function $f(v, t_f)$ can be rewritten as

$$f(v, t_f) = \sum_{i=1}^q e^{\lambda_i[\mathcal{Q}(v)]t_f}. \quad (4.5)$$

That is, $f(v, t_f)$ consists of the sum of the exponential of the eigenvalues of $\mathcal{Q}(v)$ (weighted with t_f). For symmetric $\mathcal{Q}(v)$, it is known that all eigenvalues of $\mathcal{Q}(v)$ are real. The key property associated with the penalty function $f(v, t_f)$ and the constraint $\mathcal{Q}(v) < 0$ is expressed in the following theorem.

Theorem 4.3.1

Consider a real symmetric matrix inequality constraint of the form $\mathcal{Q}(v) < 0$, $\mathcal{Q}(v) = [\mathcal{Q}(v)]^T \in R^{q \times q}$ and the penalty function $f(v, t_f)$ defined in (4.4). Then the following statements are equivalent.

$$1. \quad \mathcal{Q}(v) < 0 \quad (4.6)$$

$$2. \quad \lim_{t_f \rightarrow \infty} f(v, t_f) = 0. \quad (4.7)$$

Proof: 1. \rightarrow 2.: If $\mathcal{Q}(v) < 0$ is satisfied then all the eigenvalues of $\mathcal{Q}(v)$ are real and negative. With (4.5) this directly implies assertion 2. Conversely, if $\lim_{t_f \rightarrow \infty} f(v, t_f) = 0$, then all eigenvalues of $\mathcal{Q}(v)$ must be negative and hence $\mathcal{Q}(v)$ is negative definite. ■

Theorem 4.3.1 shows that matrix inequalities can be enforced using the scalar cost function $f(v, t_f)$. Furthermore, in the limit as $t_f \rightarrow \infty$, $f(v, t_f)$ is zero if and only if the matrix constraint $\mathcal{Q}(v) < 0$ is satisfied. On the other hand, if $\mathcal{Q}(v)$ has at least one positive definite eigenvalue, then the cost function will be unbounded as $t_f \rightarrow \infty$. Although numerically irrelevant, it may occur that $\mathcal{Q}(v)$ has eigenvalues directly at the origin. As $\mathcal{Q}(v)$ is assumed to be a $q \times q$ matrix, $\mathcal{Q}(v)$ has at most q eigenvalues at zero. Hence for any $t_f > 0$, $f(v, t_f)$ will satisfy

$$\mathcal{Q}(v) \leq 0 \iff 0 \leq f(v, t_f) \leq q, \quad (4.8)$$

where the exact function value depends on the number of eigenvalues at zero. Conversely, for $t_f > 0$,

$$f(v, t_f) \leq 1 \iff \mathcal{Q}(v) \leq 0. \quad (4.9)$$

Furthermore, if $\mathcal{Q}(v) < 0$, then there will always be a $t_f > 0$ such that (4.9) is satisfied. These considerations are summarized in the following corollary to theorem 4.3.1.

Corollary 4.3.1

Consider a real symmetric matrix-valued function $\mathcal{Q}(v)$ as follows. $\mathcal{Q}(v) = [\mathcal{Q}(v)]^T \in \mathbb{R}^{q \times q}$ with $v \in \mathbb{R}^p$ and the penalty function $f(v, t_f)$ defined in (4.4). Then the following statements are equivalent.

$$1. \quad \mathcal{Q}(v) \leq 0 \quad (4.10)$$

$$2. \quad 0 \leq \lim_{t_f \rightarrow \infty} f(v, t_f) \leq q \quad (4.11)$$

$$(2'. \quad \lim_{t_f \rightarrow \infty} f(v, t_f) \text{ is finite}). \quad (4.12)$$

Moreover, for the strict inequality $\mathcal{Q}(v) < 0$ there always exists a large, but finite, $t_f > 0$ such that $f(v, t_f) \leq 1$.

Clearly, in the limit as $t_f \rightarrow \infty$, $f(v, t_f)$ represents an interior point barrier function for the matrix-valued constraint $\mathcal{Q}(v) < 0$. In a practical implementation, t_f can be large but not infinite. It is easily verified that $f(v, t_f)$ is well defined for any positive finite t_f , even if the constraint $\mathcal{Q}(v) < 0$ is violated. Hence, for any positive finite t_f , $f(v, t_f)$ is an exterior point penalty function. This property is attractive since

it allows the optimization of the cost function to be performed for increasing values of t_f until the constraint is satisfied. The problem at hand is to find a v such that the maximum eigenvalue of $\mathcal{Q}(v)$ is negative. It is well known that such maximum-eigenvalue problems are in general non-smooth. In this formulation all eigenvalues contribute to the cost function. The more negative the k^{th} eigenvalue is, the smaller is its contribution to the overall cost. Assume that the maximum eigenvalue $\bar{\lambda}[\mathcal{Q}(v)]$ of $\mathcal{Q}(v)$ is positive while all other eigenvalues $\lambda_k[\mathcal{Q}(v)]$ are negative. In this case t_f can be chosen such that $e^{(\bar{\lambda}[\mathcal{Q}(v)])t_f} \gg e^{(\lambda_k[\mathcal{Q}(v)])t_f}$ for all k , where $\lambda_k[\mathcal{Q}(v)] \neq \bar{\lambda}[\mathcal{Q}(v)]$. Then the desired maximum eigenvalue problem is approximated while, in the limit, as $t_f \rightarrow \infty$, only the maximum eigenvalue (or eigenvalues) will contribute to the cost function. Generally, in a numerical implementation, it is desirable to increase t_f rapidly to large values so that only the remaining positive eigenvalues of $\mathcal{Q}(v)$ are penalized.

4.3.1 Convex Matrix Inequalities

To emphasize another important property of the cost function $f(v, t_f)$ in (4.4), let us look into two alternative cost functions that can be used to compute solutions for matrix inequality constraints. Such alternative cost functions $\tilde{f}(v, t_f)$ and $\hat{f}(v, t_f)$ include – but are not restricted to – the following functions.

$$\tilde{f}(v, t_f) = \sum_{k=1}^q \tilde{f}_k(v, t_f) \quad (4.13)$$

$$\tilde{f}_k(v, t_f) = \begin{cases} (e^{\{\lambda_k[\mathcal{Q}(v)]t_f\}} - 1)^2 & \text{if } \lambda_k[\mathcal{Q}(v)] \geq 0 \\ 0 & \text{if } \lambda_k[\mathcal{Q}(v)] < 0, \end{cases} \quad (4.14)$$

and

$$\hat{f}(v, t_f) = \sum_{k=1}^q \hat{f}_k(v, t_f) \quad (4.15)$$

$$\hat{f}_k(v, t_f) = \begin{cases} t_f (\lambda_k[\mathcal{Q}(v)])^2 & \text{if } \lambda_k[\mathcal{Q}(v)] \geq 0 \\ 0 & \text{if } \lambda_k[\mathcal{Q}(v)] < 0. \end{cases} \quad (4.16)$$

Due to the symmetry assumption on $\mathcal{Q}(v)$, $\mathcal{Q}(v)$ can be diagonalized for any v and hence $\tilde{f}(v, t_f)$ and $\hat{f}(v, t_f)$ are continuous and differentiable in v as long as $\mathcal{Q}(v)$ is continuous and differentiable in v . In this formulation the k^{th} eigenvalue of $\mathcal{Q}(v)$

contributes to the overall cost function only if it violates the desired constraint. In a numerical implementation these alternate cost functions will be superior to $f(v, t_f)$ for small t_f . The reader is reminded that in $f(v, t_f)$ all eigenvalues will contribute to the cost function, regardless of their sign. Eigenvalues that satisfy the inequality will be negligible in $f(v, t_f)$ only if t_f is large. However, $f(v, t_f)$ has one advantage, that both $\tilde{f}(v, t_f)$ and $\hat{f}(v, t_f)$ do not share. Namely, they are not convex in the considered parameters v , even if the underlying matrix function $\mathcal{Q}(v)$ is convex. Conditions for a scalar function to be convex are well known. This concept has a direct equivalence for symmetric matrix functions.

Definition 4.3.1

Consider a real symmetric matrix-valued function $\mathcal{Q}(v)$ of the form $\mathcal{Q}(v) = [\mathcal{Q}(v)]^T \in R^{q \times q}$ with $v \in R^p$. Let $v_1 \in R^p$, $v_2 \in R^p$ and $\alpha \in [0, 1]$, then $\mathcal{Q}(v)$ is a convex matrix function if

$$\mathcal{Q}[\alpha v_1 + (1 - \alpha)v_2] \leq \alpha \mathcal{Q}(v_1) + (1 - \alpha)\mathcal{Q}(v_2), \quad (4.17)$$

where convexity is defined in terms of the usual ordering of symmetric matrices.

With this definition we can state a very important property of $f(v, t_f)$ in the following theorem.

Theorem 4.3.2

Consider a real symmetric convex matrix-valued function $\mathcal{Q}(v)$ with $\mathcal{Q}(v) = [\mathcal{Q}(v)]^T \in R^{q \times q}$ and $v \in R^p$. Then, for a given t_f , $f(v, t_f)$ is convex in v .

Proof: Since $\mathcal{Q}(v)$ is convex in v , there is a positive semi-definite matrix Q such that

$$\mathcal{Q}[\alpha v_1 + (1 - \alpha)v_2] = \alpha \mathcal{Q}(v_1) + (1 - \alpha)\mathcal{Q}(v_2) + Q. \quad (4.18)$$

Then the following chain of equalities and inequalities prove theorem 4.3.2.

$$f[v_1 + (1 - \alpha)v_2, t_f] = \text{Trace}\{e^{\mathcal{Q}[\alpha v_1 + (1 - \alpha)v_2]t_f}\} \quad (4.19)$$

$$= \text{Trace}\{e^{[\alpha \mathcal{Q}(v_1) + (1 - \alpha)\mathcal{Q}(v_2) + Q]t_f}\} \quad (4.20)$$

$$\leq \text{Trace}\{e^{[\alpha \mathcal{Q}(v_1) + (1 - \alpha)\mathcal{Q}(v_2)]t_f}\} \quad (4.21)$$

$$\leq [\text{Trace}\{e^{\mathcal{Q}(v_1)t_f}\}]^\alpha [\text{Trace}\{e^{\mathcal{Q}(v_2)t_f}\}]^{(1 - \alpha)} \quad (4.22)$$

$$\leq \alpha \text{Trace}\{e^{\mathcal{Q}(v_1)t_f}\} + (1 - \alpha) \text{Trace}\{e^{\mathcal{Q}(v_2)t_f}\} \quad (4.23)$$

$$\leq \alpha f(v_1, t_f) + (1 - \alpha)f(v_2, t_f). \quad (4.24)$$

Here we have used equation (4.18) to show the equality of (4.19) and (4.20). Equation (4.21) follows immediately from Weyl's theorem (appendix A), the monotonicity property of eigenvalues of hermitian matrices and the fact that the (scalar) exponential function is a (strictly) monotonic function of its argument. Equation (4.22) is a direct consequence of lemma A.1.6 in appendix A. The final result follows from the arithmetic-geometric mean inequality (lemma A.1.5 in appendix A), applied to equation (4.22). This concludes the proof of theorem 4.3.2. ■

Note that this convexity property does not depend on $Q(v) < 0$ and hence is valid even if the desired inequality constraint is violated. This property is clearly illustrated in the scalar case. Let us consider $f(v, t_f) = e^{vt_f}$ for a real scalar v . Obviously $\frac{\partial^2 f(v, t_f)}{\partial v^2} = t_f^2 e^{vt_f} \geq 0$ for any real scalar v regardless of the sign of v and hence convexity follows. This is a very powerful result as it allows the representation of *convex* matrix inequalities as *convex* differentiable constraints in an overall gradient-based optimization scheme. This is of particular importance as more and more control problems are defined in terms of convex (linear) matrix inequalities ([15]).

4.3.2 Gradient Computation

To be able to utilize the defined cost function in an efficient gradient-based scheme, it is important to have explicit gradient expressions available. The derivation of such gradient expressions is presented in this section. A first important result is the fact that if $Q(v)$ is continuous and differentiable in v , then so is $f(v, t_f)$. Assuming continuity of each entry of $Q(v)$, a standard result from perturbation theory states that the eigenvalues of $Q(v)$ are continuous in v . Hence, by use of equation (4.5) and the fact that the (scalar) exponential of a function is a continuous, strictly monotonic function of its argument, continuity of $f(v, t_f)$ in v follows immediately. Differentiability in this context refers to the component-wise differentiability of the (i, j) -th entry of $[Q(v)]_{ij}$ with respect to individual components v^k , $k = 1, 2, \dots, p$ of v . With the symmetry assumption on $Q(v)$, $Q(v)$ is diagonalizable for any v and hence gradients for $\lambda_i[Q(v)]$ are well defined (see e.g. [67]). By use of equation (4.5) it is easily shown that $f(v, t_f)$ is differentiable with respect to v as well. Henceforth it is assumed that $Q(v)$ is continuous and differentiable with respect to v . Note that in the above discussion no restriction has been placed on the form of v . A repre-

sensation of v in $\mathcal{Q}(v)$ may in general be a scalar, a vector or a matrix. Explicit gradient expressions depend on the considered matrix inequality. However, a general form for these gradient expressions can be derived using a power series expansion of the matrix exponential in $f(v, t_f)$. With the differentiability of $\mathcal{Q}(v)$ at a point v_o , a linearization of $\mathcal{Q}(v)$ at v_o can proceed as follows.

$$\mathcal{Q}(v_o + dv) = \mathcal{Q}(v_o) + d\mathcal{Q}(v_o, dv) + r(dv) \quad (4.25)$$

where $d\mathcal{Q}(v_o, dv)$ is the variation of $\mathcal{Q}(v)$ at v_o due to a variation of dv around v_o . $d\mathcal{Q}(v_o, dv)$ is linear in dv such that $d\mathcal{Q}(v_o, dv) = 0$ for $dv = 0$ and $r(dv)$ collects all higher-order terms in dv . Disregarding the term $r(dv)$, the power series expansion of $\mathcal{Q}(v_o + dv)$ becomes

$$f(v_o + dv, t_f) = \text{Trace}\{e^{[\mathcal{Q}(v_o + dv)]t_f}\} \quad (4.26)$$

$$= \text{Trace}\{e^{[\mathcal{Q}(v_o) + d\mathcal{Q}(v_o, dv)]t_f}\} \quad (4.27)$$

$$= \text{Trace}\left\{\sum_{k=0}^{\infty} \frac{t_f^k}{k!} [\mathcal{Q}(v_o) + d\mathcal{Q}(v_o, dv)]^k\right\} \quad (4.28)$$

$$= \text{Trace}\left\{I + \sum_{l=1}^{\infty} \left[\frac{[\mathcal{Q}(v_o)]^l}{l!} + l t_f \frac{[\mathcal{Q}(v_o)]^{l-1}}{l!} d\mathcal{Q}(v_o, dv)\right]\right\} \quad (4.29)$$

$$= \text{Trace}\{e^{[\mathcal{Q}(v_o)]t_f}\} + t_f \text{Trace}\{e^{[\mathcal{Q}(v_o)]t_f} d\mathcal{Q}(v_o, dv)\} \quad (4.30)$$

$$= f(v_o, t_f) + t_f \text{Trace}\{e^{[\mathcal{Q}(v_o)]t_f} d\mathcal{Q}(v_o, dv)\}. \quad (4.31)$$

Going from equation (4.28) to (4.29) higher-order terms in dv have been neglected and the property $\text{Trace}(LM) = \text{Trace}(ML)$ for any compatible matrices M and L has been used. Hence we have derived the following expression.

$$f(v_o + dv, t_f) - f(v_o, t_f) = t_f \text{Trace}\{e^{[\mathcal{Q}(v_o)]t_f} d\mathcal{Q}(v_o, dv)\}. \quad (4.32)$$

If we can express $t_f \text{Trace}\{e^{[\mathcal{Q}(v_o)]t_f} d\mathcal{Q}(v_o, dv)\}$ as

$$t_f \text{Trace}\{e^{[\mathcal{Q}(v_o)]t_f} d\mathcal{Q}(v_o, dv)\} = t_f \text{Trace}\{\mathcal{R}(v_o, t_f) dv\}, \quad (4.33)$$

then, according to Kleinman's lemma ([133], appendix A) we have

$$\frac{\partial f(v, t_f)}{\partial v} \Big|_{v=v_o} = t_f [\mathcal{R}(v_o, t_f)]^T. \quad (4.34)$$

Once again, this derivation is independent of the particular representation of v . That is, in the above derivation v may be a scalar, a vector or a matrix. This result allows the computation of explicit closed-form gradient expressions for a variety of matrix inequality constraints. As $Q(v)$ is symmetric and hence diagonalizable, gradient expressions can also be expressed in terms of the individual eigenvalues and corresponding eigenvectors of $Q(v)$ using (4.5). This procedure is illustrated further in the appendices.

For the cost function $f(v, t_f) = \text{Trace}\{e^{[Q(v)]t_f}\}$ it is obvious that, if the desired inequality constraint $Q(v) < 0$ is satisfied, then $f(v, t_f) \rightarrow 0$ in the limit $t_f \rightarrow \infty$. The question arises if this is the case for the gradients as well. The answer is affirmative. This is a very important fact for the optimization process to be applied to this problem. To illustrate this property consider the scalar problem $Q(v) = v^1$ for a scalar v^1 . The corresponding inequality constraint under consideration is then $v^1 < 0$ and the cost function in (4.4) has now the following form: $f(v, t_f) = e^{v^1 t_f}$. The trace operator does not have to be applied in this case. The gradient for this case is easily computed to be

$$\frac{\partial e^{v^1 t_f}}{\partial v^1} = t_f e^{v^1 t_f}. \quad (4.35)$$

If the inequality is violated, the gradient expression (4.35) will obviously go to infinity in the limit as $t_f \rightarrow \infty$. On the other hand, if $v^1 < 0$ is satisfied, then L'Hospital's rule can be applied to show the following.

$$\lim_{t_f \rightarrow \infty} t_f e^{v^1 t_f} = \lim_{t_f \rightarrow \infty} \frac{t_f}{e^{-v^1 t_f}} \quad (4.36)$$

$$= \lim_{t_f \rightarrow \infty} \frac{1}{-v^1 e^{-v^1 t_f}} \quad (4.37)$$

$$= 0 \quad (4.38)$$

where we have used $-v^1 > 0$ and hence $\lim_{t_f \rightarrow \infty} -v^1 e^{-v^1 t_f} = \infty$. In general it can be shown that $t_f e^{v^1 t_f}$ as a function of t_f has minima at $t_f = 0$ and $t_f = \infty$ and a maximum at $t_f = \frac{1}{|v^1|}$ for $v^1 < 0$. Equivalent conclusions can be drawn for the general matrix case as well. With the assumption that $Q(v)$ is not a function of t_f and the representation of $f(v, t_f)$ in terms of the eigenvalues of $Q(v)$ in (4.5), the above limiting argument for the general matrix case can be reduced to the scalar

case. With $v = \{v^1, v^2, \dots, v^p\}$ and

$$f(v, t_f) = \sum_{i=1}^q e^{\lambda_i[\mathcal{Q}(v)]t_f}$$

the partial gradient with respect to one component of v , say v^k , is computed from

$$\frac{\partial f(v, t_f)}{\partial v^k} = \sum_{i=1}^q t_f \frac{\partial \lambda_i[\mathcal{Q}(v)]}{\partial v^k} e^{\lambda_i[\mathcal{Q}(v)]t_f}. \quad (4.39)$$

As $\frac{\partial \lambda_i[\mathcal{Q}(v)]}{\partial v^k}$ is not a function of t_f , the above argument is directly applicable to show that

$$\lim_{t_f \rightarrow \infty} \frac{\partial f(v, t_f)}{\partial v^k} = 0 \quad (4.40)$$

if all $\lambda_i[\mathcal{Q}(v)] < 0$. Hence, the cost function defined in (4.4) not only has the property that the cost function value will diminish, but the gradients tend to zero as well when the considered inequality constraint is satisfied and $t_f \rightarrow \infty$.

Note that the block-structured matrix inequalities (2.28) can also be handled with this scheme. In this case the particular block-structure has to be exploited when forming $\mathcal{R}(v_o, t_f)$ in equation (4.33). For example, consider a constraint $\mathcal{Q}(v) < 0$ where $\mathcal{Q}(v)$ has the following 2×2 block structure,

$$\mathcal{Q}(v) = \begin{pmatrix} \mathcal{Q}_{11}(v) & \mathcal{Q}_{12}(v) \\ \mathcal{Q}_{12}^T(v) & \mathcal{Q}_{22}(v) \end{pmatrix} = [\mathcal{Q}(v)]^T. \quad (4.41)$$

The corresponding structure for the matrix exponential of $\mathcal{Q}(v)$ is

$$e^{[\mathcal{Q}(v)]t_f} = \mathcal{E}(v) = \begin{pmatrix} \mathcal{E}_{11}(v) & \mathcal{E}_{12}(v) \\ \mathcal{E}_{12}^T(v) & \mathcal{E}_{22}(v) \end{pmatrix}. \quad (4.42)$$

Then equation (4.32) for this block-structured constraint is as follows.

$$\begin{aligned} f(v_o + dv, t_f) - f(v_o, t_f) &= t_f \text{Trace} \{ \mathcal{E}_{11}(v_o) d\mathcal{Q}_{11}(v_o, dv) + \mathcal{E}_{12}(v_o) d\mathcal{Q}_{12}^T(v_o, dv) \\ &\quad + \mathcal{E}_{12}^T(v_o) d\mathcal{Q}_{12}(v_o, dv) + \mathcal{E}_{22}(v_o) d\mathcal{Q}_{22}(v_o, dv) \}. \end{aligned} \quad (4.43)$$

Following the derivation in (4.33) and (4.34), the gradient expressions will now depend on the corresponding block structure of $\mathcal{Q}(v)$ and $\mathcal{E}(v)$ respectively. This scheme is easily extended to matrix constraints with more complicated block structures. This implies that gradients can be computed for any block-structured matrix inequality $\mathcal{Q}(v)$ as long as $\mathcal{Q}(v)$ is continuous in the parameters v . Thus the cost function can be applied to enforce LMI-constraints according to equations (2.28), (2.29) or (3.34)–(3.36). Explicit gradient expressions for the ARI's representing \mathcal{H}_∞ -constraints are included in appendix B.

Convex (and especially linear) matrix inequalities form a very special class. For matrix inequalities $\mathcal{Q}(v)$ that are affine and hence linear in v , gradient expressions for $f(v, t_f)$ can always be derived explicitly. Let v^1, v^2, \dots, v^p denote the individual real scalar elements of v . For the class of affine matrix inequalities, we may write $\mathcal{Q}(v)$ in a standardized form.

$$\mathcal{Q}(v) = Q^0 + v^1 Q^1 + v^2 Q^2 + \dots + v^p Q^p \quad (4.44)$$

$$= Q^0 + \sum_{k=1}^p v^k Q^k \quad (4.45)$$

where Q^p and Q^0 are constant matrices. With $\mathcal{Q}(v)$ given in (4.45), partial derivatives with respect to v^i are simply

$$\frac{\partial f(v, t_f)}{\partial v^i} = t_f \text{Trace}\{e^{\mathcal{Q}(v)t_f} Q^i\}. \quad (4.46)$$

Although numerically interesting, this formulation has a drawback. In most inequality constraints the scalars v^i will represent the elements of some matrix (e.g. a controller representation C_0). Explicit gradient expressions as a function of this matrix often provide important structural information for the problem at hand. This information of the matrix constraint in its original form may be “hidden” in this standardized form. This will be seen later, when gradient expressions for various ARI's are analyzed. Even for the standardized form it is difficult to obtain explicit closed-form second-order gradients. A characterization in terms of infinite series is possible ([67]). However, using the identity (A.80) in appendix A and the tools for finite-time cost function gradients in [64] it appears to be possible to derive computable expressions for the second-order gradients. Future research along these lines should prove valuable.

4.4 Summary

1. In this chapter a new scalar cost function has been defined that allows the substitution of real symmetric matrix inequalities by a scalar constraint in an overall optimization context. For finite t_f this function plays a role as an exterior point penalty function while in the limit, as $t_f \rightarrow \infty$, $f(v, t_f)$ becomes an interior point barrier function for the matrix inequality constraint. Unlike the formulation in [15] (see equation (4.3)), this cost function is a true barrier function in the limit as $t_f \rightarrow \infty$.

2. Let $v_1 \in R^{n_1}$, $v_2 \in R^{n_2}$, ..., $v_{n_p} \in R^{n_p}$, then multiple matrix constraints of the form

$$[\mathcal{Q}_1(v_1) < 0, \mathcal{Q}_2(v_2) < 0, \dots, \mathcal{Q}_{n_p}(v_{n_p}) < 0] \quad (4.47)$$

are easily transformed into n_p scalar constraints

$$\begin{aligned} \lim_{t_f \rightarrow \infty} \text{Trace}\{e^{[\mathcal{Q}_1(v_1)]t_f}\} &= 0, \\ \lim_{t_f \rightarrow \infty} \text{Trace}\{e^{[\mathcal{Q}_2(v_2)]t_f}\} &= 0, \\ &: \\ \lim_{t_f \rightarrow \infty} \text{Trace}\{e^{[\mathcal{Q}_{n_p}(v_{n_p})]t_f}\} &= 0. \end{aligned} \quad (4.48)$$

This framework allows the incorporation of multiple \mathcal{H}_∞ -constraints into *any* performance optimization problem. It is known that the central controller is unstable if certain subsystems of the open-loop plant have right-half plane zeros. In the presented formulation the requirement for a stable controller is easily incorporated by additional matrix constraints such as $A_c^T Y + Y A_c < 0$ and $Y = Y^T > 0$. Eigenvalue constraints on the closed-loop system can be added in the form of inequalities $[A_{cl} + \alpha I]^T Y + Y [A_{cl} + \alpha I] < 0$ and $Y = Y^T > 0$.

3. The technique is applicable to \mathcal{H}_∞ -problems in the continuous-time domain as well as in the discrete-time domain in terms of ARI's or block-structured LMI's (see appendix D).
4. If $\mathcal{Q}(v)$ is differentiable with respect to v , then $f(v, t_f)$ is also differentiable. Explicit (closed-form) gradient expressions have been derived according to the scheme outlined in equations (4.25) through (4.34). For many types of matrix

constraints the gradient computation involves only a matrix exponential and elementary matrix computations such as matrix multiplication and addition. Hence gradients can be computed very efficiently.

5. If the underlying matrix inequality is not convex in the considered parameters, then the cost functions $\tilde{f}(v, t_f)$ and $\hat{f}(v, t_f)$ in (4.13) and (4.15) respectively can be used for the problem at hand. However, if the matrix function $\mathcal{Q}(v)$ is convex, then, for a given t_f , the scalar cost $f(v, t_f)$ is also convex. In general $\tilde{f}(v, t_f)$ and $\hat{f}(v, t_f)$ will not be convex. Convexity of $f(v, t_f)$ for convex $\mathcal{Q}(v)$ is a very important feature of the cost function in (4.4) which allows its application to a large class of important control problems.

Chapter 5

A NEW APPROACH TO \mathcal{H}_∞ -SYNTHESIS

In this chapter we will concentrate only on the \mathcal{H}_∞ -design problem. With the cost function defined in (4.4) it is obvious that the corresponding \mathcal{H}_∞ -constraints in the overall $\mathcal{H}_2/\mathcal{H}_\infty$ -design problem are merely a set of scalar constraints added to the \mathcal{H}_2 -optimization problem. Hence it is important to numerically and theoretically analyze the pure \mathcal{H}_∞ -problem in this formulation before applying it to the overall mixed strategy. The extension to the mixed performance/stability robustness problem will follow naturally from these considerations.

5.1 Multi-Plant \mathcal{H}_∞ -Design Problem in Terms of a Scalar Cost Function

For the pure \mathcal{H}_∞ -problem inherent to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design, we assume $w_2^i(t) = 0$ for $i = 1, 2, \dots, n_p$ in the open-loop systems given in (3.1). The n_p closed-loop systems subject to \mathcal{H}_∞ -constraints are

$$\Sigma_{\infty,cl}^i(C_0) : \begin{cases} \dot{x}_\infty^i(t) &= A_{cl}^i x_\infty^i(t) + B_{cl,\infty}^i w_\infty^i(t) \\ z_\infty^i(t) &= C_{cl,\infty}^i x_\infty^i(t) + D_{cl,\infty}^i w_\infty^i(t) \end{cases} \quad (5.1)$$

with

$$\begin{aligned} A_{cl}^i &= \bar{A}^i + \bar{B}_3^i C_0 \bar{C}_3^i \\ B_{cl,\infty}^i &= \bar{B}_2^i + \bar{B}_3^i C_0 \bar{D}_{32}^i \\ C_{cl,\infty}^i &= \bar{C}_2^i + \bar{D}_{23}^i C_0 \bar{C}_3^i \\ D_{cl,\infty}^i &= \bar{D}_{22}^i + \bar{D}_{23}^i C_0 \bar{D}_{32}^i \end{aligned} \quad (5.2)$$

where all the relevant matrices have been defined in chapter 3. With the cost function $f(v, t_f)$ defined in (4.4) we can now rephrase definition 3.4.1 for the pure \mathcal{H}_∞ -design problem. Note that we consider at this point the suboptimal \mathcal{H}_∞ -design problem and hence γ^i for $i = 1, 2, \dots, n_p$ are assumed to be specified a-priori (a possible \mathcal{H}_∞ -optimal design strategy will be discussed in section 5.4). In this case one searches for a controller that satisfies all the n_p \mathcal{H}_∞ -constraints $\|T_\infty^i(C_0)\|_\infty < \gamma^i$ simultaneously.

Definition 5.1.1

Consider n_p closed-loop plant conditions given in (5.1) satisfying the assumptions A1 through A4. Then the \mathcal{H}_∞ -design problem for the general multi-plant output-feedback case can be posed in terms of $f(v, t_f)$ as follows. Find a controller C_0 and a set of n_p matrices X^i such that the following constraints are satisfied:

$$\begin{aligned}
 1.) \quad & \lim_{t_{f1}^i \rightarrow \infty} f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i) = 0 \\
 2.) \quad & \lim_{t_{f2}^i \rightarrow \infty} f_D^i(C_0, \gamma^i, t_{f2}^i) = 0 \\
 3.) \quad & \lim_{t_{f3}^i \rightarrow \infty} f_X^i(X^i, t_{f3}^i) = 0 \\
 4.) \quad & X^i = (X^i)^T
 \end{aligned} \tag{5.3}$$

with

$$f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i) = \text{Trace} \{ e^{ARI_{C,OF}^i(C_0, X^i, \gamma^i) t_{f1}^i} \} \tag{5.4}$$

$$f_D^i(C_0, \gamma^i, t_{f2}^i) = \text{Trace} \{ e^{-R^i t_{f2}^i} \} \tag{5.5}$$

$$f_X^i(X^i, t_{f3}^i) = \text{Trace} \{ e^{-X^i t_{f3}^i} \} \tag{5.6}$$

and

$$ARI_{C,OF}^i(C_0, X^i, \gamma^i) = [A_{cl}^i + B_{cl,\infty}^i (R^i)^{-1} (D_{cl,\infty}^i)^T C_{cl,\infty}^i]^T X^i \tag{5.7}$$

$$+ X^i [A_{cl}^i + B_{cl,\infty}^i (R^i)^{-1} (D_{cl,\infty}^i)^T C_{cl,\infty}^i]$$

$$+ X^i B_{cl,\infty}^i (R^i)^{-1} (B_{cl,\infty}^i)^T X^i + (\gamma^i)^2 (C_{cl,\infty}^i)^T (S^i)^{-1} C_{cl,\infty}^i$$

$$R^i = (\gamma^i)^2 I - (D_{cl,\infty}^i)^T D_{cl,\infty}^i \tag{5.8}$$

$$S^i = (\gamma^i)^2 I - D_{cl,\infty}^i (D_{cl,\infty}^i)^T \tag{5.9}$$

with n_p preselected robust stability bounds γ^i and a set of real positive scalars t_{fk}^i ($k = 1, 2, 3$ and $i = 1, 2, \dots, n_p$).

The real positive scalars t_{fk}^i in this formulation are used as scaling variables to condition the numerical behavior of the algorithm as discussed below. Constraint 2 in (5.3) ensures that $\bar{\sigma}(D_{cl,\infty}^i) < \gamma^i$ and forms a necessary condition for $\|T_\infty^i(C_0)\|_\infty < \gamma^i$. Constraint 1 imposes the actual ARI-matrix inequality constraint for the \mathcal{H}_∞ -bound. Of course, the expression for $ARI_{C,OF}^i(C_0, X^i, \gamma^i)$ in (5.7) for the i^{th} plant condition

can be replaced by any of the equivalent characterizations described in chapter 2. The constraints 3 and 4 in (5.3) impose the symmetry and positivity constraints on all X^i . Note that all $D_{cl,\infty}^i$ and hence all R^i and S^i are generally functions of the controller parameters in C_0 . For notational simplicity this dependency is omitted in the remainder of this report. For the further discussion the following sets are defined.

$$\mathcal{X} := \{ X^i : X^i = (X^i)^T, i = 1, 2, \dots, n_p \} \quad (5.10)$$

$$\mathcal{T}_f := \{ t_{f1}^1, t_{f1}^2, \dots, t_{f1}^{n_p}, t_{f2}^1, t_{f2}^2, \dots, t_{f2}^{n_p}, t_{f3}^1, t_{f3}^2, \dots, t_{f3}^{n_p} \} \quad (5.11)$$

$$\mathcal{G} := \{ \gamma^1, \gamma^2, \dots, \gamma^{n_p} \}. \quad (5.12)$$

The set \mathcal{X} collects the n_p symmetric matrices X^i in the above problem description, \mathcal{T}_f is the set of all scaling factors t_{fk}^i ($k = 1, 2, 3$ and $i = 1, 2, \dots, n_p$) and \mathcal{G} collects all the prespecified \mathcal{H}_∞ -bounds γ^i . With a slight abuse of notation the expressions “ $\lim_{\mathcal{T}_f \rightarrow \infty}$ ” and “ $\mathcal{T}_f \rightarrow \infty$ ” mean that the individual elements t_{fk}^i of \mathcal{T}_f approach infinity and do not imply an infinite number of elements in \mathcal{T}_f . The expression “finite \mathcal{T}_f ” will refer to elements in \mathcal{T}_f being finite and not to the number of elements in \mathcal{T}_f .

5.2 Analysis of the Individual Cost Functions

5.2.1 Algorithm Outline

The formulation of the multi-plant suboptimal problem with the cost function defined in chapter 4 makes the solution readily amenable to gradient-based parameter optimization methods. Definition 5.1.1 leads to an iterative scheme where the set \mathcal{T}_f can be used to properly scale the cost functions. Depending on the initial guesses and the maximum eigenvalues of the matrix expressions $ARI_{C,OF}^i(C_0, X^i, \gamma^i)$, $[(\gamma^i)^2 I - (D_{cl,\infty}^i)^T D_{cl,\infty}^i]$ and $-X^i$ in definition 5.1.1, an initial set \mathcal{T}_f^o can be chosen such that the cost functions are numerically well defined and have reasonable values in their gradients. Starting with \mathcal{T}_f^o a numerical optimization is performed to minimize the corresponding cost functions in an attempt to solve all the relevant constraints. After the optimization has converged for \mathcal{T}_f^o , the individual elements in \mathcal{T}_f^o are increased to form \mathcal{T}_f^1 and the optimization process is continued. This process is repeated until all the relevant constraints are satisfied.

Definition 5.1.1 only states a set of constraints and their functional representation in terms of a scalar cost function. Hence the problem formulation leads in general

a multi-objective optimization problem. This remaining question is how to solve the underlying optimization problem at the k^{th} iteration. One can usually pose the problem as a constrained minimization problem by choosing one of the constraint cost functions in definition 5.1.1 as the cost to be minimized, say $f_{ARI_{C,OF}}^1(C_0, X^1, \gamma^1, t_{f1}^1)$, subject to the remaining constraints in definition 5.1.1. Alternatively one can form a single cost function equal to the sum of all the relevant constraint cost functions over all the plant conditions and define an unconstrained optimization problem based on this single cost function. This latter approach has been adopted in this work.

Before this numerical approach is presented in more detail, it is important to analyze the individual cost functions in terms of continuity, minimally necessary number of optimization variables, existence of local minima and so forth. Particularly important in this analysis is the assurance that the overall gradients are truly zero only if the desired constraints are satisfied. Using an optimization strategy based on increasing elements in \mathcal{T}_f , these issues will be explored in more detail in the following sections.

5.2.2 Continuity of the Constraint Cost Functions

Continuity and smoothness of the cost functions are important for the convergence of gradient-based optimization methods. It is clear that R^i and $-X^i$ are continuous matrix functions in the design variables C_0 and X^i . Given any finite-valued set \mathcal{T}_f , this implies that $f_D^i(C_0, \gamma^i, t_{f2}^i)$ and $f_X^i(X^i, t_{f3}^i)$ are also continuous for all the design plant conditions and design parameters. The matrix function $ARI_{C,OF}^i(C_0, X^i, \gamma^i)$ on the other hand becomes discontinuous in the controller parameters C_0 at values where $|R^i| = 0$, that is at points where a singular value of $D_{cl,\infty}^i$ equals γ^i . This is due to the fact that the inverse of R^i appears in the matrix expression $ARI_{C,OF}^i(C_0, X^i, \gamma^i)$. The occurrence of such parameter combinations has to be avoided throughout the optimization or else the gradient-based search would become ill-conditioned. Hence in the following two different types of constraints have to be considered:

1. Continuity constraints: $R^i > 0$ for $i = 1, 2, \dots, n_p$. These constraints have to be satisfied throughout the optimization. In particular an initial controller guess has to satisfy all of these constraints before the optimization process can be started.

2. Necessary constraints for the \mathcal{H}_∞ -bounds: All the other constraints and their functional representation in definition 5.1.1, i.e. $X^i > 0$ and $ARI_{C,OF}^i(C_0, X^i, \gamma^i) < 0$. These constraints may be violated in intermediate phases of the optimization.

In section 5.3 it will be shown how these constraints are actually enforced in a practical optimization. Note, however, that such continuity problems can be avoided if block structured LMI-characterizations such as (2.28) or (2.29) are chosen to represent the \mathcal{H}_∞ -constraints. In these representations the terms R^i do not appear as inverses. However, these representations will require a larger computational effort as the matrix exponentials that have to be computed in the function and gradient computations are of larger dimensions.

5.2.3 The Symmetry Requirement for X^i and the Number of Optimization Variables

The number of optimization variables n_{con} associated with the controller parametrization C_0 is fixed by the choice of the controller structure and order. If n_{x^i} is the order of the i^{th} plant, and n_c is the specified controller order, then due to the symmetry requirement for X^i we clearly do not have to use $(n_{x^i} + n_c)^2$ optimization variables to represent X^i in the optimization process. In general one can define a set of upper triangular matrices \tilde{X}^i , ($i = 1, 2, \dots, n_p$) as actual optimization variables with

$$\tilde{X}^i = \begin{pmatrix} \tilde{X}_{1,1}^i & \tilde{X}_{1,2}^i & \tilde{X}_{1,3}^i & \dots & \tilde{X}_{1,(n_{x^i}+n_c-1)}^i & \tilde{X}_{1,(n_{x^i}+n_c)}^i \\ 0 & \tilde{X}_{2,2}^i & \tilde{X}_{2,3}^i & \dots & \tilde{X}_{2,(n_{x^i}+n_c-1)}^i & \tilde{X}_{2,(n_{x^i}+n_c)}^i \\ 0 & 0 & \tilde{X}_{3,3}^i & \dots & \tilde{X}_{3,(n_{x^i}+n_c-1)}^i & \tilde{X}_{3,(n_{x^i}+n_c)}^i \\ \vdots & & & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \tilde{X}_{(n_{x^i}+n_c-1),(n_{x^i}+n_c-1)}^i & \tilde{X}_{(n_{x^i}+n_c-1),(n_{x^i}+n_c)}^i \\ 0 & 0 & 0 & \dots & 0 & \tilde{X}_{(n_{x^i}+n_c),(n_{x^i}+n_c)}^i \end{pmatrix} \quad (5.13)$$

which requires only

$$n_X^i = \frac{1}{2}(n_{x^i} + n_c)(n_{x^i} + n_c + 1) \quad (5.14)$$

design variables per plant condition to represent X^i . The desired symmetric solution X^i for $ARI_{C,OF}^i(C_0, X^i, \gamma^i)$ is formed from \tilde{X}^i as follows: $X^i = \tilde{X}^i + (\tilde{X}^i)^T - \text{diag}[\tilde{X}^i]$, where the additional term $\text{diag}[\tilde{X}^i]$ is introduced for technical reasons to conform with

the notational convention for function gradient computations with respect to symmetric matrices (see appendix A). Gradients with respect to \tilde{X}^i are easily computed from the gradients with respect to X^i in appendix A. If X^i is factorized in this way, it is immediate that $X^i = (X^i)^T$, but $X^i > 0$ is not necessarily satisfied.

An alternative factorization of X^i in terms of \tilde{X}^i that explicitly imposes both symmetry and positive semi-definiteness of X^i ($X^i \geq 0$), can be defined by considering \tilde{X}^i as a Cholesky factor of X^i . That is, $X^i = (\tilde{X}^i)^T \tilde{X}^i$, which is symmetric and positive semi-definite for any \tilde{X}^i . Strict positive definiteness can be incorporated by adding $\varepsilon_x^i I$ to this expression for a set of small positive ε_x^i ($i = 1, 2, \dots, n_p$) to form $X^i = \varepsilon_x^i I + (\tilde{X}^i)^T \tilde{X}^i$ which is guaranteed to be positive definite and symmetric. Note that under this factorization the constraint 4 in definition 5.1.1 would no longer be required. Moreover, gradients of all the cost functions with respect to \tilde{X}^i can also be derived from the results with respect to X^i as shown in appendix B. For $X^i = (\tilde{X}^i)^T \tilde{X}^i$, gradients of $f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)$ and $f_X^i(X^i, t_{f3}^i)$ with respect to \tilde{X}^i are

$$\frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial \tilde{X}^i} = 2\tilde{X}^i \frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial X^i} \quad (5.15)$$

$$\frac{\partial f_X^i(X^i, t_{f3}^i)}{\partial \tilde{X}^i} = 2\tilde{X}^i \frac{\partial f_X^i(X^i, t_{f3}^i)}{\partial X^i}. \quad (5.16)$$

Clearly, gradient expressions for the actual optimization variable \tilde{X}^i can be easily obtained from the original gradients with respect to X^i .

Although the constraint $X^i > 0$ need not necessarily be imposed if $X^i = \varepsilon_x^i I + (\tilde{X}^i)^T \tilde{X}^i$, it is immediate from (5.15) and (5.16) that $\tilde{X}^i = 0$ would be a sufficient condition for all partial gradients with respect to \tilde{X}^i to vanish. If with $\tilde{X}^i = 0$ the gradients with respect to C_0 also vanish, then the optimization will be “stuck” in a local minimum that does not necessarily satisfy the inequality constraints necessary for the \mathcal{H}_∞ -constraints. Even when the gradients with respect to C_0 are not zero, this case may lead to numerically ill-conditioned situations where the C_0 -gradients are large in comparison to those with respect to \tilde{X}^i . Numerical experimentation has shown that these cases can indeed occur. Hence, for numerical reasons it is necessary to impose the constraint $X^i > 0$ for the case where $X^i = (\tilde{X}^i)^T \tilde{X}^i$ and thus the two parametrizations are deemed equivalent.

In the following discussion we do not necessarily use the representation of X^i in terms of \tilde{X}^i , as this does not alter the conclusions and would significantly complicate the overall discussion. When a particular form of X^i is considered, then this will be stated explicitly. It should be kept in mind, however, that the actual optimization is performed over a set of n_p upper triangular matrices \tilde{X}^i as above and hence the overall number n_{var} of optimization variables is

$$n_{var} = n_{con} + \frac{1}{2} \sum_{i=1}^{n_p} (n_{x^i} + n_c)(n_{x^i} + n_c + 1). \quad (5.17)$$

5.2.4 Analysis of the Gradient Expressions

Gradients for the cost functions in definition 5.1.1 have been derived in appendix B, using matrix results presented in appendix A. In chapter 4 it has been shown that, in the limit as $\mathcal{T}_f \rightarrow \infty$, the cost functions and their corresponding gradients will be zero if the corresponding inequality constraints in definition 5.1.1 are satisfied. It is well known that there is an infinite number of possible controllers for the suboptimal \mathcal{H}_∞ -problem – if such a solution exists. In numerical terms this implies that there is an infinite number of controllers C_0 and solutions X^i ($i = 1, 2, \dots, n_p$) that satisfy the constraints in definition 5.1.1. It is precisely this reason that makes the formulation of the \mathcal{H}_∞ -problem in terms of ARI's so attractive for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -problem. The non-uniqueness of the suboptimal \mathcal{H}_∞ -controllers can be utilized to satisfy additional constraints or – as in this work – to minimize the \mathcal{H}_2 -norm or other performance measure of some –possibly different– systems. Regardless of how the optimization problem is presented to a nonlinear optimizer, a set of possible solutions that satisfy the necessary constraints, would represent acceptable “local minima” for the suboptimal \mathcal{H}_∞ -control problem.

Note that it may be possible that all the partial gradients vanish while at the same time one or more inequality constraints are still being violated, i.e. undesirable “local minima” that do not satisfy the design goal. In general these situations have to be avoided. This concern is examined here in more detail based on the gradient information. With the proposed iterative algorithm, the analysis of these expressions is restricted to a fixed set \mathcal{T}_f . Furthermore, only the i^{th} plant condition will be considered since the same conclusions and discussion would apply to all the other plant conditions as well.

The gradients of the cost functions (5.5) and (5.6) are well behaved and become zero if and only if the corresponding constraints in (5.3) are satisfied. Numerical difficulties, however, have been encountered and are usually related to the cost function associated with the ARI-constraint in (5.4). This will be examined in the following section.

Assume the parametrization of X^i to be $X^i = \tilde{X}^i + (\tilde{X}^i)^T - \text{diag}[\tilde{X}^i]$. Without loss of generality we examine the gradients with respect to X^i since in this case the partial gradients with respect to \tilde{X}^i are zero if and only if the gradients with respect to X^i vanish. In this case the partial gradients of

$$f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i) = \text{Trace} \{ e^{ARI_{C,OF}^i(C_0, X^i, \gamma^i) t_{f1}^i} \}$$

with respect to C_0 and X^i for the i^{th} plant condition have been derived in appendix B and are as follows.

$$\begin{aligned} \frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial C_0} &= 2t_{f1}^i \{ [\bar{C}_3^i + \bar{D}_{32}^i (R^i)^{-1} (P_{aux}^i)^T] E_{ARI_{C,OF}}^i \\ &\quad [X^i \bar{B}_3^i + ((C_{cl,\infty}^i)^T + P_{aux}^i (R^i)^{-1} (D_{cl,\infty}^i)^T) \bar{D}_{23}^i] \}^T \end{aligned} \quad (5.18)$$

$$\begin{aligned} \frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial X^i} &= t_{f1}^i \{ E_{ARI_{C,OF}}^i (A_{aux}^i + B_{aux}^i X^i)^T \\ &\quad + (A_{aux}^i + B_{aux}^i X^i) E_{ARI_{C,OF}}^i \} \end{aligned} \quad (5.19)$$

with

$$ARI_{C,OF}^i(C_0, X^i, \gamma^i) = (A_{aux}^i)^T X^i + X^i A_{aux}^i + X^i B_{aux}^i X^i + C_{aux}^i \quad (5.20)$$

$$R^i = (\gamma^i)^2 I - (D_{cl,\infty}^i)^T D_{cl,\infty}^i \quad (5.21)$$

$$S^i = (\gamma^i)^2 I - D_{cl,\infty}^i (D_{cl,\infty}^i)^T \quad (5.22)$$

$$A_{aux}^i = A_{cl}^i + B_{cl,\infty}^i (R^i)^{-1} (D_{cl,\infty}^i)^T C_{cl,\infty}^i \quad (5.23)$$

$$B_{aux}^i = B_{cl,\infty}^i (R^i)^{-1} (B_{cl,\infty}^i)^T \quad (5.24)$$

$$C_{aux}^i = (\gamma^i)^2 (C_{cl,\infty}^i)^T (S^i)^{-1} C_{cl,\infty}^i \quad (5.25)$$

$$P_{aux}^i = X^i B_{cl,\infty}^i + (C_{cl,\infty}^i)^T D_{cl,\infty}^i \quad (5.26)$$

$$E_{ARI_{C,OF}}^i = e^{ARI_{C,OF}^i(C_0, X^i, \gamma^i) t_{f1}^i} \quad (5.27)$$

Note at this point that both expressions (5.18) and (5.19) contain one common matrix, namely $t_{f1}^i E_{ARI_{C,OF}}^i$. Furthermore, this matrix enters both expressions multi-

plicatively such that

$$\frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial C_0} = 0 \quad \text{and} \quad \frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial X^i} = 0$$

if $t_{f1}^i E_{ARI_{C,OF}}^i = 0$. Applying the limiting argument in chapter 4 (see (4.36 – (4.38)) to this matrix expression, it can be verified that all partial gradients are indeed zero in the limit as $t_{f1}^i \rightarrow \infty$, if the i^{th} ARI-constraint $ARI_{C,OF}^i(C_0, X^i, \gamma^i) < 0$ is satisfied. These solutions are therefore acceptable. Possible non-acceptable local minima satisfying

$$\begin{aligned} \frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial C_0} &= 0 \\ \frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial X^i} &= 0 \end{aligned}$$

with

$$ARI_{C,OF}^i(C_0, X^i, \gamma^i) \not\leq 0$$

are the topic of the following analysis.

Let us assume that the inequality $ARI_{C,OF}^i(C_0, X^i, \gamma^i) < 0$ is violated in at least one mode. First note that $E_{ARI_{C,OF}}^i$ is a positive definite matrix for any finite t_{f1}^i and $ARI_{C,OF}^i(C_0, X^i, \gamma^i)$. In this case sufficient conditions for all the partial gradients to be zero are found from (5.18) and (5.19) and are as follows.

$$A_{aux}^i + B_{aux}^i X^i = 0, \quad (5.28)$$

and
either

$$\bar{C}_3^i + \bar{D}_{32}^i (R^i)^{-1} (P_{aux}^i)^T = 0 \quad (5.29)$$

or

$$X^i \bar{B}_3^i + [(C_{cl,\infty}^i)^T + P_{aux}^i (R^i)^{-1} (D_{cl,\infty}^i)^T] \bar{D}_{23}^i = 0. \quad (5.30)$$

Here (5.28) causes $\frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial X^i} = 0$ and (5.29) or (5.30) are sufficient conditions for $\frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial C_0} = 0$. These equations immediately identify solutions that have to be avoided. If either (5.28) and (5.29) or (5.28) and (5.30) are satisfied then all respective gradients are zero and the algorithm is “stuck” at this particular point. Note that the gradients will remain zero even as t_{f1}^i is increased. Hence the iterative optimization procedure breaks down and will not converge.

For the general dynamic output–feedback case, X^i will have the following structure

$$X^i = \begin{pmatrix} X_{11}^i & X_{12}^i \\ (X_{12}^i)^T & X_{22}^i \end{pmatrix}, \quad (5.31)$$

where the individual blocks of X^i are assumed to be of dimensions compatible to the internal block structure of the relevant closed–loop matrices (see (3.17) – (3.18)). Utilizing this internal structure of X^i it can be shown after some tedious algebra that the partial gradients with respect to C_0 can be zero only if $X_{22}^i = 0$. However, this would violate the assumption that X^i is positive definite and hence the constraint 3 in definition 5.1.1. This fact follows immediately from the Schur complement formula (see appendix A). Thus, for the general dynamic output–feedback case with $X^i > 0$, the overall gradients with respect to C_0 will be zero only if the i^{th} ARI–constraint is satisfied. This is not necessarily true for the static output–feedback case where X^i reduces to $X^i = X_{11}^i$. In this case there may be C_0 and X^i such that (5.29) or (5.30) are satisfied.

The partial gradients with respect to X^i on the other hand may be zero regardless of the considered case. Possible solutions for $A_{aux}^i + B_{aux}^i X^i = 0$ can be defined in terms of the Moore Penrose inverse of B_{aux}^i as follows (see e.g. [67]).

$$A_{aux}^i + B_{aux}^i X^i = 0 \iff X^i = (X^i)^T = -(B_{aux}^i)^+ A_{aux}^i + [I - (B_{aux}^i)^+ B_{aux}^i] Z^i \quad (5.32)$$

for an arbitrary matrix Z^i of compatible dimensions.

Although in a numerical implementation this case will not occur exactly, numerical experimentation has shown that these situations typically lead to slow convergence and numerically small X^i –gradients in comparison to those associated with C_0 . To gain additional insight into this situation let us assume that, during an optimization, the following case has occurred. The inequality $ARI_{C,OF}^i(C_0, X^i, \gamma^i) < 0$ is violated

in at least one mode, A_{cl}^i and A_{aux}^i are both asymptotically stable, $R^i > 0$ and $X^i > 0$ such that $A_{aux}^i + B_{aux}^i X^i = 0$ for a given controller C_0 . With $X^i > 0$ the expression $A_{aux}^i + B_{aux}^i X^i = 0$ is equivalent to the following identities.

$$X^i A_{aux}^i + X^i B_{aux}^i X^i = 0 \quad (5.33)$$

$$A_{aux}^i (X^i)^{-1} + B_{aux}^i = 0. \quad (5.34)$$

Note that (5.33) and (5.34) are also true for the respective transposes. Substituting (5.34) into (2.35) in theorem 2.2.1 yields

$$[(X^i)^{-1} - \hat{L}_c^i](A_{aux}^i)^T + A_{aux}^i[(X^i)^{-1} - \hat{L}_c^i] + B_{aux}^i = 0 \quad (5.35)$$

where \hat{L}_c solves

$$\hat{L}_c^i (A_{aux}^i)^T + A_{aux}^i \hat{L}_c^i + B_{aux}^i = 0. \quad (5.36)$$

With A_{aux}^i asymptotically stable and $B_{aux}^i \geq 0$ this implies that $[(X^i)^{-1} - \hat{L}_c^i] \geq 0$ (see statement 6 in theorem 2.2.1) for $ARI_{C,OF}^i(C_0, X^i, \gamma^i) < 0$ is not violated. A similar argument shows that requirement 5 in theorem 2.2.1 is satisfied as well.

By subtracting (5.33) and its transpose (both are assumed to be zero) from equation (2.33) in theorem 2.2.1 it can be shown that for $A_{aux}^i + B_{aux}^i X^i = 0$ the following identity is valid.

$$(A_{cl}^i)^T [L_o^i - X^i] + [L_o^i - X^i] A_{cl}^i + C_{aux}^i - X^i B_{aux}^i X^i - P_{aux}^i (R^i)^{-1} (P_{aux}^i)^T = 0 \quad (5.37)$$

where L_o^i solves

$$(A_{cl}^i)^T L_o^i + L_o^i A_{cl}^i + (C_{cl}^i)^T C_{cl}^i = 0. \quad (5.38)$$

Using a standard Lyapunov argument with the assumption that A_{cl}^i is asymptotically stable, we cannot conclude that $L_o^i - X^i \leq 0$ as required in theorem 2.2.1. Whether or not this constraint is satisfied, will depend on whether the matrix $C_{aux}^i - X^i B_{aux}^i X^i - P_{aux}^i (R^i)^{-1} (P_{aux}^i)^T$ in (5.37) is negative definite or not. From the given information one is not able to confirm this property. In general this implies that X^i may be “too small” when $A_{aux}^i + B_{aux}^i X^i = 0$ and thereby could violate the necessary condition $X^i \geq L_o^i$ (see theorem 2.2.1) for $ARI_{C,OF}^i(C_0, X^i, \gamma^i) < 0$. Similar conclusions can be drawn for the cases when A_{cl}^i and/or A_{aux}^i are not asymptotically stable. An analogous analysis of the alternative form of $ARI_{C,OF}^i(C_0, X^i, \gamma^i)$ in (2.32) suggests that X^i may be “too large”.

The above observations give rise to possible additional constraints to better condition the numerical optimization scheme. Such additional constraints could be

$$\lim_{t_{f4}^i \rightarrow \infty} \text{Trace}\{e^{[L_o^i - X^i]t_{f4}^i}\} = 0 \quad (5.39)$$

$$\lim_{t_{f5}^i \rightarrow \infty} \text{Trace}\{e^{[\hat{L}_c^i - (X^i)^{-1}]t_{f5}^i}\} = 0 \quad (5.40)$$

where \hat{L}_c^i and L_o^i solve (5.36) and (5.38) respectively, subject to the additional stability constraints on A_{cl}^i and A_{aux}^i . Gradient expressions for these functions and the stability constraints can be derived using the formalism presented in appendix A for functions involving grammians and eigenvalue constraints. The stability constraint can also be imposed via a Lyapunov-type inequality and cost functions such as (4.13) or (4.15). Such additional constraints have been tested numerically and have been found to improve the overall performance of the algorithm. However, no analytical proof has been found to show that these constraints are sufficient conditions for $A_{aux}^i + B_{aux}^i X^i \neq 0$ and hence cannot in general resolve the gradient-related problem discussed above.

In summary, there are possible situations where all partial gradients become zero without the relevant constraints being satisfied. In this case, the iterative algorithm proposed above will fail to converge to a solution that satisfies the desired \mathcal{H}_∞ -constraints. The possible occurrence of such situations is independent of the particular realization of X^i . The only identifiable violations of necessary conditions for $ARI_{C,OF}^i(C_0, X^i, \gamma^i) < 0$ are given by the fact that X^i may be “too small”, violating condition 3 in theorem 2.2.1 or “too large”. Additional constraints such as (5.39) or (5.40) have improved the numerical behavior of the problem but do not in general guarantee that all the respective partial gradients are non-zero when the set \mathcal{T}_f is updated in the iterative scheme. It is suspected that there are analytical interrelations between the system assumptions in [24] and [113] and the corresponding necessary conditions for “local minima” as discussed above. However, without any further assumptions on the sign-definiteness of $ARI_{C,OF}^i(C_0, X^i, \gamma^i)$ no theoretical basis has been found for this conjecture. Future research will have to show if such interconnections exist. In short, the formulation for the suboptimal \mathcal{H}_∞ -controller design in definition 5.1.1, when solved via parameter optimization methods, may not deliver the desired results, and modifications have to be made to account for these cases.

Considering the above iterative algorithm, any possible modification needs to satisfy the following criteria. Assume that a solution has been found such that, for a given set \mathcal{T}_f the cost functions in definition 5.1.1 are minimized and their respective gradients are zero. If one or more inequality constraints are not satisfied, then the corresponding partial gradients with respect to at least C_0 or X^i must be non-zero as the elements of \mathcal{T}_f are increased. This requirement ensures that the iterative algorithm outlined above, will converge to a desired solution that satisfies all the \mathcal{H}_∞ -constraints and eliminates local minima that do not satisfy these constraints. In the following, various perturbation approaches are presented to address this problem.

From chapter 2 it is known that whenever the i^{th} \mathcal{H}_∞ -constraint is satisfied, then there is a X^i such that $ARI_{C,OF}^i(C_0, X^i, \gamma^i) < 0$ and $A_{aux}^i + B_{aux}^i X^i$ is stable. Hence an obvious choice for a possible modification is to impose the additional constraints

$$[A_{aux}^i + B_{aux}^i X^i]Y^i + Y^i[A_{aux}^i + B_{aux}^i X^i]^T < 0 \quad (5.41)$$

$$Y^i = (Y^i)^T > 0 \quad (5.42)$$

for a set of n_p matrices Y of compatible dimensions to directly impose stability of $A_{aux}^i + B_{aux}^i X^i$ for all the plant conditions. This would imply that the gradients with respect to X^i will not be zero after a \mathcal{T}_f update. However, this approach will require an additional set of $\frac{1}{2}(n_{x^i} + n_c)(n_{x^i} + n_c + 1)$ optimization variables per plant condition. In general these stability constraints can be represented by constraint functions defined directly in terms of the eigenvalues of $A_{aux}^i + B_{aux}^i X^i$ as in (4.13) or (4.15). Such a formulation would not require additional optimization variables but is in general not differentiable for non-symmetric matrices such as $A_{aux}^i + B_{aux}^i X^i$.

Alternatively one can solve the so-called “ ζ -shifted \mathcal{H}_∞ -problem” (see e.g. [13]) by introducing n_p scalar perturbation parameters $\zeta^i \geq 0$ for which the following modified ARI-constraints with cost functions in definition 5.1.1 are formulated for $i = 1, 2, \dots, n_p$.

$$ARI_{C,OF}^i(C_0, X^i, \gamma^i, \zeta^i) < 0 \quad (5.43)$$

$$ARI_{C,OF}^i(C_0, X^i, \gamma^i, \zeta^i) := [(A_{aux}^i)^T + \zeta^i I]X^i + X^i[A_{aux}^i + \zeta^i I] + X^i B_{aux}^i X^i + C_{aux}^i. \quad (5.44)$$

Note that for $\zeta^i = 0$ the original \mathcal{H}_∞ -problem is recovered. This modified problem imposes the additional constraints that all the eigenvalues of A_{cl}^i have real parts smaller than $-\zeta^i$. Gradients of the corresponding cost functions

$$f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, \zeta^i, t_{f1}^i) = \text{Trace} \{ e^{ARI_{C,OF}^i(C_0, X^i, \gamma^i, \zeta^i) t_{f1}^i} \} \quad (5.45)$$

are easily found by substituting

$$A_{aux}^i \rightarrow A_{aux}^i + \zeta^i I \quad (5.46)$$

in the gradient expressions for the cost function (5.4). These ζ^i can be used to take possible occurrences of $A_{aux}^i + B_{aux}^i X^i = 0$ into account which now read $A_{aux}^i + \zeta^i I + B_{aux}^i X^i = 0$ for the new cost function. Starting with an initial set ζ^i , $i = 1, 2, \dots, n_p$, after each iteration and update of \mathcal{T}_f , ζ^i can be adjusted so as to make $A_{aux}^i + \zeta^i I + B_{aux}^i X^i \neq 0$. Typically one would start out with a prespecified value for all the ζ^i , say 1, which is reduced as t_{f1}^i increases. In the limit, as $t_{f1}^i \rightarrow \infty$ the original, unconstrained \mathcal{H}_∞ -problem is solved using such an update rule. In this case the overall gradients with respect to X^i will be zero if and only if $ARI_{C,OF}^i(C_0, X^i, \gamma^i, \zeta^i) < 0$ which implies that the desired \mathcal{H}_∞ -constraint and the additional eigenvalue constraint on A_{cl}^i are satisfied. This formulation has worked well in the numerical implementation but has a drawback in so far as there may be situations where the partial gradients with respect to C_0 may still be zero while the partial gradients with respect to X_i are large.

Another modification to circumvent the problem of local minima is to use an auxiliary cost and a set of perturbation parameters. Consider the modified cost functions $\bar{f}_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)$ as follows.

$$\begin{aligned} \bar{f}_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i) \\ = f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i) + c^i \nu^i(t_{f1}^i) \{ \text{Trace}[(X^i)^T X^i + (C_0)^T C_0] \} \end{aligned} \quad (5.47)$$

with

$$\nu^i(t_{f1}^i) = \begin{cases} 1 & \text{if } 0 < t_{f1}^i \leq 1 \\ \frac{1}{t_{f1}^i} & \text{if } 1 < t_{f1}^i \end{cases} \quad (5.48)$$

$$c^i \leq (t_{f1}^i)^\kappa, \quad 0 \leq \kappa < 1 \quad (5.49)$$

for $i = 1, 2, \dots, n_p$. A typical value for κ is $\kappa = 0.5$ and hence $c^i \leq \sqrt{t_{f1}^i}$. With $c^i \leq (t_{f1}^i)^\kappa$, $0 \leq \kappa < 1$ it follows immediately that

$$0 \leq \lim_{t_{f1}^i \rightarrow \infty} c^i \nu^i(t_{f1}^i) \leq \lim_{t_{f1}^i \rightarrow \infty} \frac{(t_{f1}^i)^\kappa}{t_{f1}^i} \quad (5.50)$$

$$\leq \lim_{t_{f1}^i \rightarrow \infty} \frac{1}{(t_{f1}^i)^{1-\kappa}} \quad (5.51)$$

$$\leq 0 \quad (5.52)$$

as $1 - \kappa > 0$ for $i = 1, 2, \dots, n_p$. Hence it follows that

$$\lim_{t_{f1}^i \rightarrow \infty} \bar{f}_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i) = f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i). \quad (5.53)$$

Thus in the limit as $t_{f1}^i \rightarrow \infty$, the original cost functions are recovered. Note that the problem formulation in definition 5.1.1 does not take into account the possible occurrence of unbounded C_0 and X^i . One desirable side effect of the above modification is that for any finite t_{f1}^i the cost $\bar{f}_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)$ will be finite if and only if C_0 and X^i are finite. Hence finiteness requirements on C_0 and X^i are directly incorporated into the cost function. Moreover, it is well known that the auxiliary term $\text{Trace}[(X^i)^T X^i + (C_0)^T C_0]$ is convex in C_0 and X^i (see lemma A.1.6) and hence possible convexity properties of the original problem are maintained as the sum of convex functions is convex. The scalars c^i allow the weighting of $c^i \nu^i(t_{f1}^i) \{\text{Trace}[(X^i)^T X^i + (C_0)^T C_0]\}$ such that $f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i) > c^i \nu^i(t_{f1}^i) \{\text{Trace}[(X^i)^T X^i + (C_0)^T C_0]\}$. Let us now analyze how these additional terms alter the gradient expressions. The gradient expressions for the auxiliary cost $\text{Trace}[(X^i)^T X^i + (C_0)^T C_0]$ are well known and the overall gradients for the new cost function $\bar{f}_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)$ are as follows.

$$\frac{\partial \bar{f}_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial X^i} = \frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial X^i} + 2c^i \nu^i(t_{f1}^i) X^i \quad (5.54)$$

$$\frac{\partial \bar{f}_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial C_0} = \frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial C_0} + 2c^i \nu^i(t_{f1}^i) C_0. \quad (5.55)$$

With (5.18) and (5.19), necessary conditions for $\frac{\partial \bar{f}_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial C_0} = 0$ and

$\frac{\partial \bar{f}_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial X^i} = 0$ are as follows.

$$2t_{f1}^i \{ [\bar{C}_3^i + \bar{D}_{32}^i (R^i)^{-1} (P_{aux}^i)^T] E_{ARI_{C,OF}}^i \times [X^i \bar{B}_3^i + ((C_{cl,\infty}^i)^T + P_{aux}^i (R^i)^{-1} (D_{cl,\infty}^i)^T) \bar{D}_{23}^i] \}^T + 2c^i \nu^i(t_{f1}^i) C_0 = 0 \quad (5.56)$$

and

$$t_{f1}^i \{ E_{ARI_{C,OF}}^i (A_{aux}^i + B_{aux}^i X^i)^T + (A_{aux}^i + B_{aux}^i X^i) E_{ARI_{C,OF}}^i \} + 2c^i \nu^i(t_{f1}^i) X^i = 0. \quad (5.57)$$

Now, for any finite t_{f1}^i , $\frac{\partial \bar{f}_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial C_0} = 0$, $\frac{\partial \bar{f}_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial X^i} = 0$ and $X^i > 0$ it immediately follows that

$$(A_{aux}^i + B_{aux}^i X^i) \quad \text{is asymptotically stable} \quad (5.58)$$

$$(|A_{aux}^i + B_{aux}^i X^i| \neq 0) \quad (5.59)$$

$$\bar{C}_3^i + \bar{D}_{32}^i (R^i)^{-1} (P_{aux}^i)^T \neq 0 \quad (5.60)$$

$$X^i \bar{B}_3^i + [(C_{cl,\infty}^i)^T + P_{aux}^i (R^i)^{-1} (D_{cl,\infty}^i)^T] \bar{D}_{23}^i \neq 0. \quad (5.61)$$

Note that this formulation implies asymptotic stability of $A_{aux}^i + B_{aux}^i X^i$. However, with the results in chapter 2 (see corollary 2.2.1), it is known that, if a controller satisfies the i^{th} \mathcal{H}_∞ -constraint, then there will always be a X^i that satisfies this additional requirement and hence this formulation does not represent a loss of generality. Most importantly equations (5.58) – (5.61) imply that if the partial gradients are zero for a given t_{f1}^i but the corresponding ARI-constraint is violated, a change in t_{f1}^i will result in non-zero partial gradients for both, C_0 and X^i . This in turn implies that an iterative scheme that minimizes the cost functions iteratively based on a strictly monotonically increasing sequence t_{f1}^i will not converge to local minima that do not satisfy the desired \mathcal{H}_∞ -constraints, and hence the likely occurrence of such local minima is eliminated. Thus, for a large but finite t_{f1}^i , the gradients of the modified cost function will be zero if and only if all the constraints are satisfied. Furthermore, this scheme requires only a minor modification of the cost functions in definition 5.1.1, and has worked well numerically.

Some final comments on the numerical schemes are in order. Numerical simulation has shown that direct stability constraints on $A_{aux}^i + B_{aux}^i X^i$ in combination with

the other perturbation methods is efficient. However, the additional optimization variables required to enforce these constraint in terms of a Lyapunov inequalities may be considerable and therefore numerically harder to tackle. All the above schemes either require additional optimization variables or involve a perturbation scheme. Ideally one would like to have a scheme in which zero gradients directly imply that all the constraints are satisfied for *any* finite set \mathcal{T}_f . At the same time this property should be accomplished without the introduction of additional optimization variables so as to not slow down the overall numerical computations. In general the various forms for the ARI-constraints such as (2.22) – (2.24), $(X^i)^{-1}ARI_{C,OF}(C_0, X^i, \gamma^i)(X^i)^{-1} < 0$ or their LMI-representations as well as other alternative ways to enforce $X^i > 0$ (for example $(X^i)^{-1} > 0$) may provide possible alternative modifications or additional constraints to arrive at a numerically effective scheme. Research along these lines should be pursued in the future.

5.3 Numerical Approaches and a Penalty/Barrier Function Approach to the Multi-Plant \mathcal{H}_∞ -Design Problem

In this section all the above results and the proposed iterative scheme to solve the \mathcal{H}_∞ -design problem are combined into one computational framework. In particular a penalty/barrier function approach is presented to solve the general multi-plant suboptimal \mathcal{H}_∞ -design problem.

Due to the iterative nature of the proposed algorithm the problem can be solved by a sequence of constrained optimization problems. In this case one selects one of the cost functions in definition 5.1.1, e.g. $f_{ARI_{C,OF}}^1(C_0, X^1, \gamma^1, t_{f1}^1)$ as actual cost to be minimized while all the other inequality constraints are enforced as constraints in a gradient-based optimization. If the underlying inequality constraints are not convex, one may want to redefine the inequality constraints in definition 5.1.1 in terms of cost functions such as in (4.13) or (4.15); namely in terms of the eigenvalues of the matrix expressions. As mentioned before, these types of cost functions are differentiable as long as the corresponding inequality constraint is symmetric. Gradients are readily derived from the gradients for the trace type cost function (see appendices A and B). This type of cost function has the advantage that it will be zero if the corresponding constraint is satisfied independent of a scaling factor t_f . Using a penalty cost function to represent the constraints in definition 5.1.1, this is true only in the limit as the

corresponding $t_{f1(2,3)}^i \rightarrow \infty$. This is not desirable if the optimization problems are solved via constrained minimizations as in this case constraints may be defined only as scalar inequality or equality constraints requiring an exact upper (lower) bound for the inequality constraints. For finite $t_{f1(2,3)}^i$ the exact function value of the individual cost functions is not known exactly even if the inequality constraint is satisfied and hence constant upper bounds for the cost functions (and hence the constraints) may not be defined.

Consider for example the cost function $f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)$. The only knowledge we have is that $0 < f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i) < n_{x^i} + n_c$ (see chapter 2) if the constraint $ARI_{C,OF}^i(C_0, X^i, \gamma^i) < 0$ is satisfied. However, if $ARI_{C,OF}^i(C_0, X^i, \gamma^i) < 0$ is satisfied, then all eigenvalues of $ARI_{C,OF}^i(C_0, X^i, \gamma^i)$ are negative real and hence there exists a large but finite t_{f1}^i such that $f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i) < 1$. Thus, for the solution of the optimization problem via constrained minimizations, the defined cost functions in definition 5.1.1 can be utilized as scalar inequality constraints as follows.

$$f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i) < 1 \quad (5.62)$$

$$f_D^i(C_0, \gamma^i, t_{f2}^i) < 1 \quad (5.63)$$

$$f_X^i(X^i, t_{f3}^i) < 1 \quad (5.64)$$

for any set \mathcal{T}_f . If these constraints are satisfied for $i = 1, 2, \dots, n_p$, then all the related inequality constraints are also satisfied.

In this work an alternative route has been chosen to solve the \mathcal{H}_∞ -design problem utilizing the cost functions defined in definition 5.1.1. Instead of solving a constrained optimization problem for each \mathcal{T}_f , an overall cost function $J_\infty(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f)$ is defined that includes all the individual penalty functions in definition 5.1.1. Namely

$$\begin{aligned} J_\infty(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f) = \sum_{i=1}^{n_p} [& f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i) \\ & + c^i \nu^i(t_{f1}^i) \{Trace[(X^i)^T X^i + (C_0)^T C_0]\} \\ & + f_D^i(C_0, \gamma^i, t_{f2}^i) \\ & + f_X^i(X^i, t_{f3}^i)] \end{aligned} \quad (5.65)$$

where the individual cost functions, the sets \mathcal{X} , \mathcal{G} and \mathcal{T}_f and the scalars c^i and $\nu^i(t_{f1}^i)$ have been defined previously. The dependence of $J_\infty(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f)$ on these

scalars has been omitted in this notation. They will in general be constants during the optimization process as described below. It is easily verified that $J_\infty(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f)$ has the same properties as the individual cost functions in definition 5.1.1. That is, a controller C_0 and a set \mathcal{X} satisfy all the essential constraints if and only if

$$\lim_{\mathcal{T}_f \rightarrow \infty} J_\infty(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f) = 0 \quad (5.66)$$

and hence

$$\lim_{\mathcal{T}_f \rightarrow \infty} J_\infty(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f) = 0 \Leftrightarrow \|T_\infty^i(C_0)\|_\infty < \gamma^i. \quad (5.67)$$

On the other hand, if one of the constraints is not satisfied, then in the limit as $\mathcal{T}_f \rightarrow \infty$, the overall cost function $J_\infty(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f)$ will become unbounded. With this new cost function, an unconstrained optimization problem for the multi-plant suboptimal \mathcal{H}_∞ -problem can be defined as follows.

Definition 5.3.1

Consider n_p closed-loop plant conditions as in (5.1) satisfying the assumptions A1 through A4. Then the \mathcal{H}_∞ -design problem for the multi-plant output-feedback case in terms of $J_\infty(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f)$ is as follows. Given the set \mathcal{G} of user-specified \mathcal{H}_∞ -bounds, find a controller C_0^ and a set of matrices \mathcal{X}^* that solve the following minimization problem*

$$\min_{C_0, \mathcal{X}} J_\infty(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f) \quad (5.68)$$

iteratively for monotonically increasing values $t_{f_k}^i$ ($k = 1, 2, 3$, $i = 1, 2, \dots, n_p$) in \mathcal{T}_f such that

$$\lim_{\mathcal{T}_f \rightarrow \infty} J_\infty(C_0^*, \mathcal{X}^*, \mathcal{G}, \mathcal{T}_f) = 0. \quad (5.69)$$

Hence the design problem is solved via a series of unconstrained optimization problems. Once the optimization (5.68) has converged for a fixed $(\mathcal{T}_f)^k$ at the k^{th} iteration step, the elements of $(\mathcal{T}_f)^k$ are increased to form a set $(\mathcal{T}_f)^{k+1}$ for which the optimization (5.68) is repeated and so forth. This iteration continues until all the constraints have been satisfied. If the specified \mathcal{H}_∞ -bounds are chosen too tight, then the algorithm will terminate at some iteration because no sufficient decrease in the cost function can be achieved in this iteration. Specific rules for updating the set $(\mathcal{T}_f)^k$ as well as the role of the scalars c^i and $\nu^i(t_{f_1}^i)$ in this formulation along with a possible choice of initial guesses for the controller C_0 and the set \mathcal{X} are discussed next.

5.3.1 Initial Guesses, Initialization and Iteration Update Rules

Good initial guesses are vital for any nonlinear optimization problem. For this problem, numerical simulations have shown that an \mathcal{H}_2 -optimal controller for the closed-loop system $\Sigma_{\infty,cl}^i(C_0) := [A_{cl}^i, (B_{aux}^i)^{\frac{1}{2}}, (C_{aux}^i)^{\frac{1}{2}}, 0]$ has provided a good initial guess – if such a controller with the desired structure and order is available. One method to find a structurally constrained controller is given in [64]. In [139] some connections between such a \mathcal{H}_2 -optimal controller and the resulting \mathcal{H}_∞ -norm for the closed-loop system have been investigated. Any initial controller guess, however, must satisfy $(\gamma^i)^2 I - (D_{cl,\infty}^i)^T D_{cl,\infty}^i > 0$ for $i = 1, 2, \dots, n_p$. As discussed earlier, these constraints are necessary for the optimization problem at hand to be continuous in the controller parameters and hence have to be satisfied throughout the optimization. If the initial controller guess is stabilizing one or more plant conditions, then the corresponding initial guesses of X^i can be determined by $X^i = L_o^i$, where L_o^i solves

$$(A_{cl}^i)^T L_o^i + L_o^i A_{cl}^i + (C_{cl}^i)^T C_{cl}^i = 0 \quad (5.70)$$

which implies that the initial $ARI_{C,OF}^i(C_0, X^i, \gamma^i)$ reduces to

$$ARI_{C,OF}^i(C_0, X^i, \gamma^i) = P_{aux}^i (R^i)^{-1} (P_{aux}^i)^T \quad (5.71)$$

with

$$P_{aux}^i = X^i B_{cl,\infty}^i + (C_{cl,\infty}^i)^T D_{cl,\infty}^i. \quad (5.72)$$

If the initial controller guess stabilizes the i^{th} plant and satisfies the i^{th} \mathcal{H}_∞ -constraint, then a possible initial guess for X^i is the corresponding ARE-solution that can be found with any Riccati equation solver.

If the initial controller guess does not stabilize the i^{th} plant, then any symmetric, positive definite matrix X^i will suffice. In any event, we will always be able to define an initial set \mathcal{X} of symmetric positive definite matrices X^i . The initial set \mathcal{T}_f is chosen such that the overall cost function $J_\infty(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f)$ has a finite value and the corresponding partial gradients are well defined. The auxiliary parameters c^i , $i = 1, 2, \dots, n_p$ in (5.65) are initially set to values such that the sum of the \mathcal{H}_∞ -related cost functions is larger than the auxiliary cost in $J_\infty(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f)$. The parameter κ is assumed to be preselected and is constant throughout the optimization process.

In general this additional parameter has been introduced to assure that the auxiliary cost is zero in the limit as $\mathcal{T}_f \rightarrow \infty$.

Now assume that at the k^{th} iteration a controller $(C_0)^k$ and a set $(\mathcal{X})^k$ have been found for a given fixed set $(\mathcal{T}_f)^k$ and a set of scalars $(c^i)^k$, $i = 1, 2, \dots, n_p$ such that $J_\infty((C_0)^k, (\mathcal{X})^k, \mathcal{G}, (\mathcal{T}_f)^k)$ is minimized and all partial gradients of the function $J_\infty((C_0)^k, (\mathcal{X})^k, \mathcal{G}, (\mathcal{T}_f)^k)$ are zero. The question arises how to adjust $(\mathcal{T}_f)^k \mapsto (\mathcal{T}_f)^{k+1}$ and $(c^i)^k \mapsto (c^i)^{k+1}$, $i = 1, 2, \dots, n_p$ such that the problem at the subsequent $(k+1)^{th}$ iteration is well defined. Possible update rules for $(\mathcal{T}_f)^k$ may in general follow two different strategies. Assume that some but not all of the inequality constraints have been satisfied at the k^{th} iteration. In a penalty function approach one would then increase only the elements in $(\mathcal{T}_f)^k$ that correspond to active constraints while elements in $(\mathcal{T}_f)^k$ corresponding to non-active constraints would be kept at their old value. In such a set up the individual cost functions corresponding to non-active constraints would still contribute to the overall cost function $J_\infty((C_0)^{k+1}, (\mathcal{X})^{k+1}, \mathcal{G}, (\mathcal{T}_f)^{k+1})$ in the subsequent minimization. At the same time the penalty functions corresponding to active constraints become dominating in the overall cost function and the partial gradients. Alternatively, in a barrier function approach, one can choose large values for elements in $(\mathcal{T}_f)^{k+1}$ corresponding to non-active constraints. Knowing that the individual cost functions and their gradients will be approximately zero for a sufficiently large value of the corresponding element in $(\mathcal{T}_f)^{k+1}$, this is equivalent to a Lagrange multiplier approach in which a non-active constraint is deactivated by choosing a small Lagrange multiplier for this specific constraint. In this formulation the individual cost functions associated with non-active constraints will effectively act as barrier functions rejecting any choice of controllers $(C_0)^{k+1}$ and $(\mathcal{X})^{k+1}$ that violate these constraints in the subsequent iteration.

In the following, an update scheme is presented resembling a penalty function approach. The adaption of this scheme to a barrier function approach is stated later. To keep the notational complexity to a minimum, $(\nu^i(t_{f1}^i))^k$ and $(\nu^i(t_{f1}^i))^{k+1}$ will be abbreviated by $(\nu^i)^k$ and $(\nu^i)^{k+1}$ respectively. The following update scheme for the i^{th} plant condition is one of many possible approaches one can pursue. Assuming that the k^{th} iteration has been completed successfully with a controller $(C_0)^k$ and a set $(\mathcal{X})^k$, the update for the next iteration proceeds as follows.

1. t_{f1}^i -Update:

- If the i^{th} ARI-constraint is satisfied, let $(t_{f1}^i)^{k+1} = (t_{f1}^i)^k$, otherwise let

$$(t_{f1}^i)^{k+1} = \frac{1}{\bar{\lambda}[ARI_{C,OF}^i((C_0)^k, (X^i)^k, \gamma^i)]}. \quad (5.73)$$

- Set $(\nu^i)^{k+1} = \frac{1}{(t_{f1}^i)^{k+1}}$ and find the largest $(c^i)^{k+1}$ where $0 < (c^i)^{k+1} < [(t_{f1}^i)^{k+1}]^\kappa$ such that the following conditions are satisfied:

$$\begin{aligned} f_{ARI_{C,OF}}^i((C_0)^k, (X^i)^k, \gamma^i, (t_{f1}^i)^{k+1}) \\ \geq 10(c^i)^{k+1}(\nu^i)^{k+1} \{Trace[(X^i)^k]^T (X^i)^k + [(C_0)^T]^k (C_0)^k\} \end{aligned} \quad (5.74)$$

$$\begin{aligned} \bar{\sigma} \left\{ \frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, (t_{f1}^i)^{k+1})}{\partial C_0} \right\} \Big|_{\substack{C_0=(C_0)^k \\ X^i=(X^i)^k}} \\ \geq 10 (c^i)^{k+1} (\nu^i)^{k+1} \bar{\sigma} \{(C_0)^k\} \end{aligned} \quad (5.75)$$

and, if

$$\bar{\sigma} \left\{ \frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, (t_{f1}^i)^{k+1})}{\partial X^i} \right\} \Big|_{\substack{C_0=(C_0)^k \\ X^i=(X^i)^k}} > 1, \quad (5.76)$$

then

$$\begin{aligned} \bar{\sigma} \left\{ \frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, (t_{f1}^i)^{k+1})}{\partial X^i} \right\} \Big|_{\substack{C_0=(C_0)^k \\ X^i=(X^i)^k}} \\ \geq 10 (c^i)^{k+1} (\nu^i)^{k+1} \bar{\sigma} \{(X^i)^k\}. \end{aligned} \quad (5.77)$$

2. t_{f2}^i -Update:

Select $(t_{f2}^i)^{k+1} \geq (t_{f2}^i)^k$ such that

$$(t_{f2}^i)^{k+1} = \frac{10}{|\bar{\lambda}\{[(D_{cl,\infty}^i)^T D_{cl,\infty}^i - (\gamma^i)^2 I]|_{C_0=(C_0)^k}\}|} \quad (5.78)$$

3. t_{f3}^i -Update:

If the i^{th} positivity constraint $-(X^i)^k < 0$ is violated, let

$$(t_{f3}^i)^{k+1} = \frac{3}{\lambda[-(X^i)^k]} \quad (5.79)$$

or else select $(t_{f3}^i)^{k+1}$ such that

$$(c^i)^{k+1}(\nu^i)^{k+1}(X^i)^k > (t_{f3}^i)^{k+1} e^{-(X^i)^k (t_{f3}^i)^{k+1}}. \quad (5.80)$$

First note that this scheme guarantees that all elements of \mathcal{T}_f are monotonically increasing functions of the iteration number and do not decrease. The t_{f1}^i -updates assures that the ARI-cost functions associated with the active ARI-constraints contribute more to the overall cost than those corresponding to the non-active constraints. The same is true for the t_{f3}^i -updates when the i^{th} positivity constraint $-(X^i)^k < 0$ is violated. The role of the scalars c^i and ν^i will become more clear by the analysis of the gradient expressions. These gradient expressions are readily found from the sum of the gradients of the individual cost functions.

$$\begin{aligned} \frac{\partial J_\infty(C_0, \mathcal{X}, \mathcal{G}, (\mathcal{T}_f)^{k+1})}{\partial C_0} \Big|_{\substack{C_0=(C_0)^k \\ X^i=(X^i)^k}} &= \sum_{i=1}^{n_p} \left[\frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, (t_{f1}^i)^{k+1})}{\partial C_0} \Big|_{\substack{C_0=(C_0)^k \\ X^i=(X^i)^k}} \right. \\ &\quad + \frac{\partial f_D^i(C_0, \gamma^i, (t_{f2}^i)^{k+1})}{\partial C_0} \Big|_{\substack{C_0=(C_0)^k \\ X^i=(X^i)^k}} \\ &\quad \left. + 2(c^i)^{k+1}(\nu^i)^{k+1}(C_0)^k \right] \end{aligned} \quad (5.81)$$

$$\begin{aligned} \frac{\partial J_\infty(C_0, \mathcal{X}, \mathcal{G}, (\mathcal{T}_f)^{k+1})}{\partial X^i} \Big|_{\substack{C_0=(C_0)^k \\ X^i=(X^i)^k}} &= \frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, (t_{f1}^i)^{k+1})}{\partial X^i} \Big|_{\substack{C_0=(C_0)^k \\ X^i=(X^i)^k}} \\ &\quad - (t_{f3}^i)^{k+1} e^{-(X^i)^k (t_{f3}^i)^{k+1}} + 2(c^i)^{k+1}(\nu^i)^{k+1}(X^i)^k. \end{aligned} \quad (5.82)$$

The update rules (5.74) – (5.77) assure that $(c^i)^{k+1}(\nu^i)^{k+1}\{Trace [[(X^i)^k]^T(X^i)^k + ((C_0)^T C_0)^k]\}$ do not overwhelm the other \mathcal{H}_∞ -related cost functions and their respective gradients. If $(t_{f3}^i)^{k+1}$ is chosen as indicated, then $(c^i)^{k+1}(\nu^i)^{k+1}(X^i)^k >$

$(t_{f3}^i)^{k+1} e^{-(X^i)^k (t_{f3}^i)^{k+1}}$ and hence at the beginning of the $(k+1)^{th}$ iteration the partial gradients with respect to X^i are not zero, even if $A_{aux}^i + B_{aux}^i (X^i)^{k+1} = 0$. Although the $(k+1)^{th}$ iteration may converge to a parameter combination where $-(t_{f3}^i)^{k+1} e^{-(X^i)^k (t_{f3}^i)^{k+1}}$ and $2(c^i)^{k+1} (\nu^i)^{k+1} (X^i)^k$ cancel each other, due to the term $2(c^i)^{k+1} (\nu^i)^{k+1} (C_0)^k$ in the controller gradients the overall gradients force the optimization to improve on the eigenvalues of $ARI_{C,OF}(C_0, X^i, \gamma^i, (t_{f1}^i)^{k+1})$. Note that one can also enforce $X^i > 0$ by the cost function $Trace \{e^{-(X^i)^{-1} t_{f3}^i}\}$. This implies that $A_{aux}^i + B_{aux}^i (X^i)^{k+1}$ is stable if the combined gradient with respect to X^i are zero after the $(k+1)^{th}$ iteration has been completed. This additional problem is specific to the penalty/barrier function approach and is not present if the algorithm is executed as a sequence of constrained optimization problems.

The update rule for the n_p scalars $(t_{f2}^i)^{k+1}$ is not as obvious and requires some more analysis. As discussed earlier, the n_p constraints $-R^i > 0$ are necessary constraints for the minimization problems to be smooth and hence have to be enforced throughout the proposed iterative optimization procedure. Assume that $-R^i < 0$ is satisfied for a given fixed controller C_0 and consider the following bounds for $f_D^i(C_0, \gamma^i, t_{f2}^i)$ and the maximum singular value of its gradients for the i^{th} plant condition.

$$f_D^i(C_0, \gamma^i, t_{f2}^i) = Trace \{e^{-R^i t_{f2}^i}\} \quad (5.83)$$

$$\leq n_{w_\infty} e^{\bar{\lambda}(-R^i) t_{f2}^i} \quad (5.84)$$

and

$$\bar{\sigma} \left\{ \frac{\partial f_D^i(C_0, \gamma^i, t_{f2}^i)}{\partial C_0} \right\} = \bar{\sigma} \{ 2 t_{f2}^i \bar{D}_{32} e^{-R^i t_{f2}^i} (D_{cl,\infty})^T \bar{D}_{23} \} \quad (5.85)$$

$$\leq 2 t_{f2}^i e^{\bar{\lambda}(-R^i) t_{f2}^i} \bar{\sigma} \{ \bar{D}_{32} \} \bar{\sigma} \{ \bar{D}_{23} \} \bar{\sigma} \{ D_{cl,\infty} \} \quad (5.86)$$

For a fixed controller it is easily verified that the upper bound (5.84) for the cost function $f_D^i(C_0, \gamma^i, t_{f2}^i)$ is a monotonically decreasing function of t_{f2}^i (assuming $-R^i < 0$ is satisfied). The upper bound (5.85) for the gradient expressions on the other hand exhibits an interesting behavior. It is easily verified, that (5.85) is a monotonically increasing function of t_{f2}^i for $t_{f2}^i < \frac{1}{\bar{\lambda}(-R^i)}$, it has a maximum at $t_{f2}^i = \frac{1}{\bar{\lambda}(-R^i)}$ and decreases monotonically for larger values of t_{f2}^i . Graphically this implies that (5.85) has a “hump” at $t_{f2}^i = \frac{1}{\bar{\lambda}(-R^i)}$. Selecting $(t_{f2}^i)^{k+1}$ much larger than $\frac{1}{\bar{\lambda}(-R^i)}$ in the above scheme implies that both the function value of $f_D^i(C_0, \gamma^i, t_{f2}^i)$ and its gradients will

be small at the beginning of the $(k + 1)^{th}$ iteration and would dominate the overall cost function if these constraints are violated during a line search. Numerically the above update rule for $(t_{f2}^i)^{k+1}$ implies that the cost functions $f_D^i((C_0)^{k+1}, \gamma^i, (t_{f2}^i)^{k+1})$ will act as barrier functions, effectively enforcing the continuity constraints $R^i > 0$ during the $(k + 1)^{th}$ iteration.

As mentioned above, alternative update schemes may be devised. In particular, the explicit numerical values chosen for the updates have been derived from numerical experimentation utilizing the MATLAB optimization toolbox. The presented scheme represents a mixture of penalty and barrier function approaches. For the constraints that are allowed to be violated during intermediate phases of the algorithm, the above update scheme realizes a penalty function approach. The specific choices for $(t_{f2}^i)^{k+1}$ incorporate a barrier function approach so as to guarantee continuity and hence smoothness during the subsequent optimization.

In general the above update scheme can be easily adopted to an overall barrier function approach where not only the continuity constraints are enforced via barrier function ideas, but all constraint functions corresponding to non-active matrix inequalities. If the \mathcal{H}_∞ -constraint and hence the corresponding matrix inequality is satisfied at the i^{th} plant condition, then one would choose large $(t_{f1}^i)^{k+1}$, $(t_{f2}^i)^{k+1}$ and $(t_{f3}^i)^{k+1}$ for all the penalty functions associated with this \mathcal{H}_∞ -constraint. With such an update procedure all penalty functions associated with the i^{th} \mathcal{H}_∞ -constraint will act as barrier functions in the $(k + 1)^{th}$ iteration, avoiding possible controllers $(C_0)^{k+1}$ and $(X^i)^{k+1}$ that would violate any of the non-active constraints. For a more detailed discussion of penalty/barrier function approaches the reader is referred to [63] which contains an excellent treatment of these methods.

5.4 A “Top Down” Approach and the \mathcal{H}_∞ -Optimal Design Problem

The numerical solution for the multi-plant \mathcal{H}_∞ -problem in terms of $J_\infty(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f)$, starts with initial guesses for the controller C_0 and \mathcal{X} that may not satisfy any of the constraints in definition 5.1.1. The n_p desired \mathcal{H}_∞ -bounds γ^i are given a-priori and the optimization procedure attempts to satisfy these bounds. With the introduction of n_p additional optimization variables γ_0^i and the corresponding set

$$\mathcal{G}_0 = \{ \gamma_0^1 \gamma_0^2 \dots \gamma_0^{n_p} \} \quad (5.87)$$

a “Top Down” approach can be developed as described below.

Definition 5.4.1

Consider n_p closed-loop plant conditions in (5.1) satisfying the assumptions A1 through A4, the set \mathcal{G}_0 , the extended set $\hat{\mathcal{T}}_f = \{\mathcal{T}_f, t_{f\gamma}^1, \dots, t_{f\gamma}^{n_p}\}$, where the set \mathcal{T}_f has been defined previously, and $J_\infty(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f)$ as given in definition 5.3.1. Then a “Top Down” \mathcal{H}_∞ -design algorithm for the multi-plant output-feedback case is defined as follows. Given the set \mathcal{G} of user-specified \mathcal{H}_∞ -bounds, define the following cost function

$$\hat{J}_\infty(C_0, \mathcal{X}, \mathcal{G}, \mathcal{G}_0, \hat{\mathcal{T}}_f) = J_\infty(C_0, \mathcal{X}, \mathcal{G}_0, \mathcal{T}_f) + \sum_{i=1}^{n_p} e^{[(\gamma_0^i)^2 - (\gamma^i)^2]t_{f\gamma}^i} \quad (5.88)$$

and find a controller C_0^* , a set of matrices \mathcal{X}^* and a set \mathcal{G}_0^* that solve the following minimization problem

$$\min_{C_0, \mathcal{X}, \mathcal{G}_0} \hat{J}_\infty(C_0, \mathcal{X}, \mathcal{G}, \mathcal{G}_0, \hat{\mathcal{T}}_f) \quad (5.89)$$

for increasing values of $t_{f\gamma}^i$, ($k = 1, 2, 3$, $i = 1, 2, \dots, n_p$) and $\{t_{f\gamma}^1, t_{f\gamma}^2, \dots, t_{f\gamma}^{n_p}\}$ in $\hat{\mathcal{T}}_f$ such that

$$\lim_{\hat{\mathcal{T}}_f \rightarrow \infty} \hat{J}_\infty(C_0^*, \mathcal{X}^*, \mathcal{G}, \mathcal{G}_0^*, \hat{\mathcal{T}}_f) = 0. \quad (5.90)$$

Note that in this formulation $J_\infty(C_0, \mathcal{X}, \mathcal{G}_0, \mathcal{T}_f)$ defined in (5.65) is not a function of the prespecified set \mathcal{G} , but a function of the optimization parameters γ_0^i . The (user-specified) fixed bounds γ^i appear only in the additional term

$$\sum_{i=1}^{n_p} e^{[(\gamma_0^i)^2 - (\gamma^i)^2]t_{f\gamma}^i}. \quad (5.91)$$

This implies that (5.90) is satisfied if and only if

$$\begin{aligned} ARI_{C,OF}^i(C_0, X^i, \gamma_0^i) &< 0 \\ (D_{cl,\infty}^i)^T D_{cl,\infty}^i - (\gamma_0^i)^2 I &< 0 \\ -X^i &< 0 \\ (\gamma_0^i)^2 &< (\gamma^i)^2 \end{aligned}$$

for $i = 1, 2, \dots, n_p$. This set of inequalities directly implies that

$$\|T_\infty^i(C_0)\|_\infty^2 < (\gamma_0^i)^2 \quad (5.92)$$

$$(\gamma_0^i)^2 < (\gamma^i)^2 \quad (5.93)$$

and hence all specified \mathcal{H}_∞ -constraints are satisfied if (5.90) is satisfied. Gradient expressions for $J_\infty(C_0, \mathcal{X}, \mathcal{G}_0, \mathcal{T}_f)$ with respect to $(\gamma_0^i)^2$ are included in appendix B. Note that, without loss of generality, we can optimize over the n_p scalars $(\gamma_0^i)^2$, as all relevant cost functions are functions only of $(\gamma_0^i)^2$ and $(\gamma^i)^2$ and not of γ_0^i or γ^i . Gradients of the term $\sum_{i=1}^{n_p} e^{[(\gamma_0^i)^2 - (\gamma^i)^2]t_f^i}$ on the other hand are trivial scalar gradients. This formulation has a significant advantage over the previous formulation in that it allows the definition of “good” initial guesses if a controller is available that stabilizes all plant conditions simultaneously. A possible scheme for choosing the initial guesses for C_0 , \mathcal{X} and \mathcal{G}_0 and solving the corresponding optimization problem to satisfy all desired \mathcal{H}_∞ -constraints can be formulated as follows.

1. Find *any* initial controller C_0 (with the desired structure and order) that simultaneously stabilizes all plant conditions.
2. For $i = 1, 2, \dots, n_p$ choose the initial set \mathcal{G}_0 such that

$$(\gamma_0^i)^2 > \|T_\infty^i(C_0)\|_\infty^2. \quad (5.94)$$

3. For this initial set \mathcal{G}_0 and $i = 1, 2, \dots, n_p$ determine an initial set \mathcal{X} from

$$ARI_{C,OF}^i(C_0, X^i, \gamma_0^i) + \varepsilon I = 0 \quad (5.95)$$

for some small ε and the fixed initial controller guess C_0 using a standard Riccati solver.

With these initial settings all but the n_p constraints $(\gamma_0^i)^2 < (\gamma^i)^2$ are satisfied. Hence the iterative optimization to solve this problem will have well defined initial guesses. The subsequent iterative minimization process will only have to additionally enforce the constraints (5.93). As soon as these constraints are satisfied, the iteration will terminate and all \mathcal{H}_∞ -constraints are satisfied.

This formulation also gives rise to an \mathcal{H}_∞ -optimal design algorithm. Assume that all elements in the set \mathcal{G} are zero. This implies that the optimization problem in definition 5.4.1 attempts to find a controller that guarantees an infinite stability radius for every plant condition, i.e. $\|T_\infty^i(C_0)\|_\infty^2 < 0$ for $i = 1, 2, \dots, n_p$. Hence if we apply the algorithm in definition 5.4.1 to this problem, the optimization process will

try to satisfy all the relevant constraints and additionally minimize the functional $\sum_{i=1}^{n_p} e^{(\gamma_0^i)^2 t_{f\gamma}^i}$ which amounts to a minimization of all considered \mathcal{H}_∞ -norms. This is a novel approach to solve the multi-plant \mathcal{H}_∞ -optimal design problem, although \mathcal{H}_∞ -optimal controllers often exhibit many undesirable properties.

5.5 \mathcal{H}_∞ -Design Examples

All the examples in this section have the following open-loop state-space description.

$$\begin{aligned} \dot{x}^i(t) &= A^i x^i(t) + B_2^i w_\infty^i(t) + B_3^i u^i(t) \\ z_\infty^i(t) &= C_2^i x^i(t) + D_{22}^i w_\infty^i(t) + D_{23}^i u^i(t) \\ y^i(t) &= C_3^i x^i(t) + D_{32}^i w_\infty^i(t) + D_{33}^i u^i(t). \end{aligned} \quad (5.96)$$

That is, all matrices corresponding to possible \mathcal{H}_2 -objectives are assumed to be zero. The number of design plant conditions is evident from the superscripts of the corresponding system matrices.

5.5.1 Example 1

The first example considered is taken from [88]. In accordance with the notation in (5.96) the open-loop system matrices are as follows.

$$\begin{aligned} A^1 &= \begin{pmatrix} -0.3908 & -0.4565 & 1.2657 \\ 1.4453 & -1.0491 & -1.2077 \\ -0.1288 & 0.6744 & 1.0324 \end{pmatrix}, \\ B_2^1 &= \begin{pmatrix} 0.0488 \\ 0.3608 \\ 0.3564 \end{pmatrix}, \quad B_3^1 = \begin{pmatrix} -0.4275 \\ -0.4470 \\ -0.9172 \end{pmatrix}, \\ C_2^1 &= \begin{pmatrix} 0.9420 & 0.0144 & 0.1187 \end{pmatrix}, \quad D_{22}^1 = 0, \quad D_{23}^1 = 1.3575, \\ C_3^1 &= \begin{pmatrix} -1.5567 & -1.9432 & -0.0914 \end{pmatrix}, \quad D_{32}^1 = 0.5185, \quad D_{33}^1 = 0. \end{aligned}$$

The plant is open-loop unstable and all the relevant subsystems have no invariant zeros on the $j\omega$ -axis. However, the open-loop subsystems (A, B_2, C_3, D_{32}) and

(A, B_3, C_2, D_{23}) have invariant zeros in the right-half plane. The plant also satisfies the assumptions imposed in [24] for a regular \mathcal{H}_∞ -design problem. Hence the two-Riccati equation approach to the suboptimal \mathcal{H}_∞ -problem can be used to compute the central controller. Utilizing this approach it can be verified that the smallest achievable closed-loop \mathcal{H}_∞ -norm $\|T_\infty^1(C_0)\|_\infty$ is approximately 2.1426. Hence it is known that for $\gamma^1 < 2.1426$ no \mathcal{H}_∞ -suboptimal controller exists. Using the discussed penalty function approach, a controller has been designed that satisfies a specified robustness bound. For the design the \mathcal{H}_∞ -bound $\gamma^1 = 2.2$ was chosen, which is rather close to the optimally achievable \mathcal{H}_∞ -norm. The controller to be designed is a dynamic, strictly proper full-order (third-order) controller. The following controller has been obtained. Its state-space realization is given by

$$\begin{aligned} A_c &= \begin{pmatrix} -69.52247277 & 4.09046301 & -132.24247153 \\ -159.84943026 & -13.73735121 & 92.97199999 \\ -13.99956517 & -0.63916776 & -1.78390805 \end{pmatrix}, \\ B_c &= \begin{pmatrix} 7.79934347 \\ -30.37935405 \\ -1.63224803 \end{pmatrix}, \\ C_c &= \begin{pmatrix} -39.68076598 & 0.62994741 & -45.76734663 \end{pmatrix}, \quad D_c = 0. \end{aligned}$$

Since the plant satisfies all the requirements for the two-Riccati equation approach, the central controller for $\gamma^1 = 2.2$ has been computed as well to allow a comparison between the designed controller and the central controller. The state-space realization of the central controller is as follows.

$$\begin{aligned} A_c &= \begin{pmatrix} -1.72346585 & -1.01231441 & -9.35407417 \\ 0.42846443 & -0.64560199 & -6.29074683 \\ -19.75912227 & 9.71850563 & -45.31150942 \end{pmatrix}, \\ B_c &= \begin{pmatrix} -0.29369100 \\ -0.14518215 \\ -8.24424158 \end{pmatrix}, \\ C_c &= \begin{pmatrix} 0.75994761 & 0.05747133 & 16.36721454 \end{pmatrix}, \quad D_c = 0. \end{aligned}$$

Some properties for the design and the central controller are summarized in tables 5.1 and 5.2 respectively.

Table 5.1: \mathcal{H}_∞ -design example 1: Closed-loop properties for the designed controller.

Specified γ^1 :	2.2 (6.8485 dB)
Achieved $\ T_\infty^1(C_0)\ _\infty$:	2.1987 (6.8433 dB)
Closed-loop system poles:	$\lambda_1 = -80.4958$ $\lambda_2 = -2.7017$ $\lambda_{3,4} = -0.3957 \pm 1.8170j$ $\lambda_{5,6} = -0.7312 \pm 1.1993j$
Controller poles:	$\lambda_{c1} = -85.5022$ $\lambda_{c2,3} = 0.2292 \pm 2.1625j$

Table 5.2: \mathcal{H}_∞ -design example 1: Closed-loop properties for the central controller.

Specified γ^1	2.2 (6.8485 dB)
Achieved $\ T_\infty^1(C_0)\ _\infty$:	2.1991 (6.8449 dB)
Closed-loop system poles:	$\lambda_1 = -43.2017$ $\lambda_2 = -2.5091$ $\lambda_{3,4} = -0.3991 \pm 1.7837j$ $\lambda_{5,6} = -0.7895 \pm 1.2618j$
Controller poles:	$\lambda_{c1} = -48.0839$ $\lambda_{c2,3} = 0.2017 \pm 2.1338j$

Both controllers stabilize the system and the achieved closed-loop \mathcal{H}_∞ -norms for both controllers are almost identical. It can be verified that $-0.3991 \pm 1.7837j$ and $-0.7895 \pm 1.2618j$ are zeros of the open-loop subsystems (A, B_2, C_3, D_{32}) and (A, B_3, C_2, D_{23}) and hence are necessarily poles of the closed-loop system with the central controller in the loop (see e.g. [45]). The corresponding closed-loop poles for the designed controller, namely $-0.3957 \pm 1.8170j$ and $-0.7312 \pm 1.1993j$ are close to the zeros of (A, B_2, C_3, D_{32}) and (A, B_3, C_2, D_{23}) . However, they are numerically not exactly equal. This represents a departure from the properties that the central controller is bound to exhibit. Both controllers are unstable suggesting that there may not be a stable controller that internally stabilizes the system and satisfies the specified \mathcal{H}_∞ -bound. Singular value plots for the closed-loop transfer function $T_\infty^1(C_0, s)$ from $w_\infty^1(s)$ to $z_\infty^1(s)$ for the designed controller and the central controller are included in Figures 5.1 and 5.2 respectively. These plots confirm that the specified \mathcal{H}_∞ -bound has been satisfied by both controllers.

In a second design the \mathcal{H}_∞ -bound was chosen to be $\gamma^1 = 2.16$. Using the presented penalty function approach a controller has been found that satisfies this bound with $\|T_\infty^1(C_0)\|_\infty < 2.16$.

$$A_c = \begin{pmatrix} 94.72003708 & 116.54307959 & 66.04512426 \\ -66.68926127 & -104.71656006 & 20.54296555 \\ -15.73792161 & 38.51746238 & -181.78169890 \end{pmatrix},$$

$$B_c = \begin{pmatrix} -153.54027409 \\ 107.75005408 \\ 19.22234533 \end{pmatrix}$$

$$C_c = \begin{pmatrix} 0.04202796 & -11.19723338 & 33.22247777 \end{pmatrix}, \quad D_c = 0.$$

The achieved $\|T_\infty^1(C_0)\|_\infty$ is 2.1579 (6.6806 dB) which is only about 0.01 above the optimally achievable \mathcal{H}_∞ -norm for this example. However, the large matrix entries suggest that an optimal \mathcal{H}_∞ -design would require a high-gain controller.

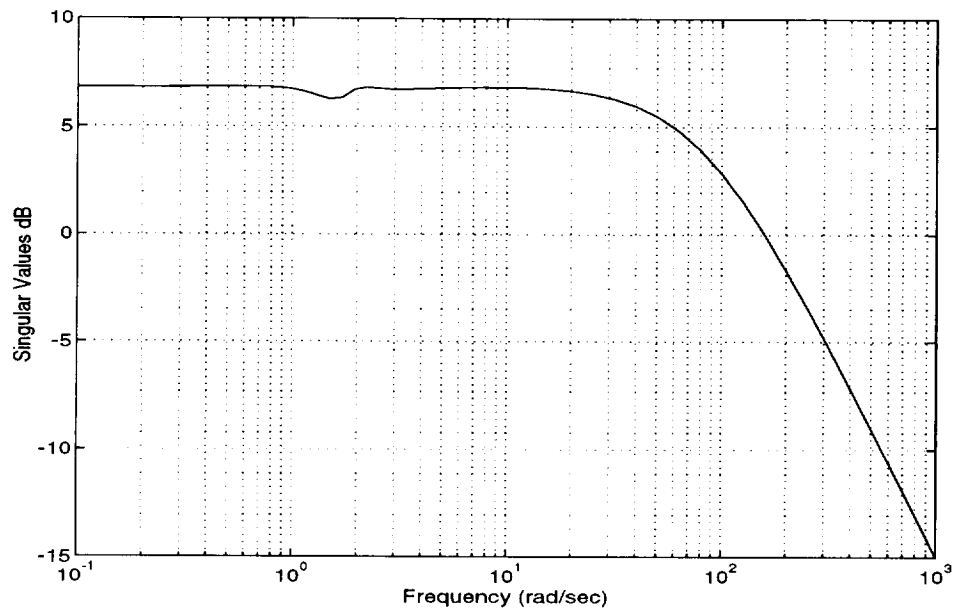


Figure 5.1: \mathcal{H}_{∞} -design example 1: Singular value plot of $T_{\infty}^1(C_0, s)$ for the designed controller.

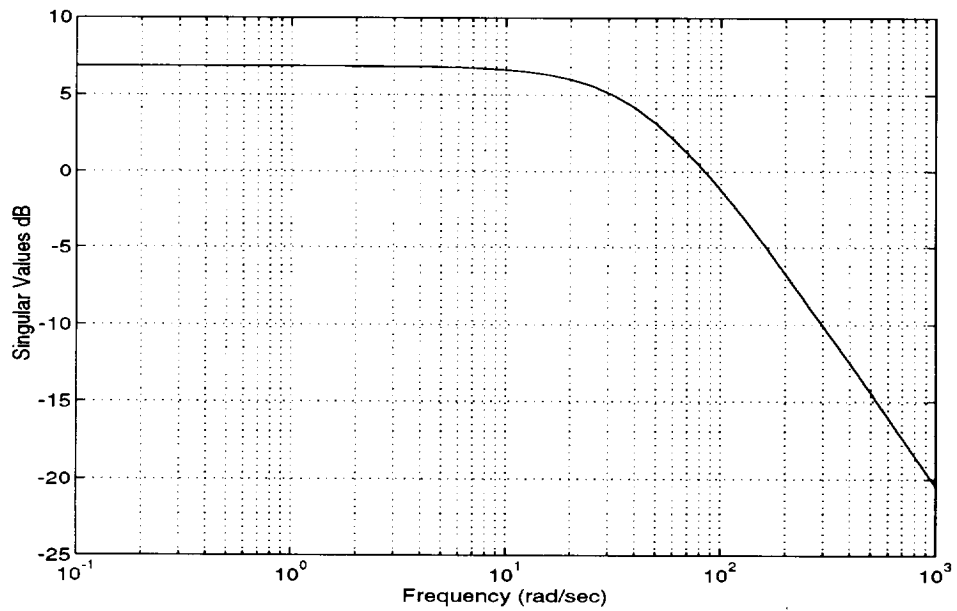


Figure 5.2: \mathcal{H}_{∞} -design example 1: Singular value plot of $T_{\infty}^1(C_0, s)$ for the central controller.

5.5.2 Example 2: The Two-Mass Spring System

The two-mass spring system has been treated extensively in recent publications. This plant was for example the benchmark problem of the 1990 American Control Conference and references describing and analyzing this plant in detail are plentiful. The open-loop system is given by

$$\begin{aligned}\dot{x}^1(t) &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m} & 0 & \frac{k}{m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{m} & 0 & -\frac{k}{m} & 0 \end{pmatrix} x^1(t) + \begin{pmatrix} 0 \\ \frac{1}{m} \\ 0 \\ -\frac{1}{m} \end{pmatrix} w_\infty^1(t) + B_3^1 u^1(t) \\ z_\infty^1(t) &= \begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix} x^1(t)\end{aligned}\quad (5.97)$$

$$y^1(t) = C_3^1 x^1(t).$$

The matrices B_3^1 and C_3^1 are specified later depending on the particular design problem under consideration. This system represents the dynamical model of two masses that are connected with a spring. In this description both masses are assumed to have the same mass m . The parameter k in (5.97) represents the spring constant of the spring connecting the two masses. The nominal values for these two parameters are $m_0 = 1$ and $k_0 = 1$. The states $x_1^1(t)$ and $x_3^1(t)$ represent the position of the first and second mass respectively. The states $x_2^1(t)$ and $x_4^1(t)$ are the corresponding velocities of the two masses. For this design example it is assumed that both masses assume the nominal value $m = m_0 = 1$. With this assumption and the matrices B_2^1 and C_2^1 in (5.97) the system description takes into account possible uncertainties in the spring constant k . Let the true spring constant k be denoted by $k = k_0 + \Delta k$ where Δk is the perturbation of k around k_0 . Then it is easily verified that

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m_0} & 0 & \frac{k}{m_0} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_0} & 0 & -\frac{k}{m_0} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_0}{m_0} & 0 & \frac{k_0}{m_0} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_0}{m_0} & 0 & -\frac{k_0}{m_0} & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m_0} \\ 0 \\ -\frac{1}{m_0} \end{pmatrix} \Delta k \begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix}\quad (5.98)$$

and hence

$$A^1 = A_{nom}^1 + B_2^1 \Delta k C_2^1 \quad (5.99)$$

where A_{nom}^1 is the nominal open-loop system matrix for $m = m_0 = 1$ and $k = k_0 = 1$. With the discussion in chapter 2 it can be verified that the uncertainty block $\Delta(s)^1$ is identical to Δk ; namely $\Delta(s)^1 = \Delta k$. Hence the open-loop system for the \mathcal{H}_∞ -design problem with Δk as uncertainty is the system (5.97) at the nominal plant condition $m = m_0 = 1$ and $k = k_0 = 1$. For this uncertainty various design cases are now examined. The open-loop system is unstable, the suboptimal \mathcal{H}_∞ -problem is singular and the subsystem (A, B_2, C_3, D_{32}) has zeros at $s = 0$ and hence on the $j\omega$ -axis, regardless of the choices for B_3^1 and C_3^1 . Thus neither the DGKF-approach nor the Stoorvogel approach are applicable to this problem. Depending on the matrices B_3^1 and C_3^1 and hence on the information available to the controller and the allowed actuation through B_3^1 , three different design situations are considered in the following subsections.

5.5.2.1 Case 1: Measurements of $x_1^1(t)$, $x_3^1(t)$ and Actuation on the First Mass

For this design case both mass positions are available to the controller while a force may be applied only to the first mass. Hence the corresponding matrices B_3^1 and C_3^1 are as follows.

$$B_3^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad C_3^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

For this design case it follows immediately by inspection, that only positive values for k can be tolerated. That is, “active” springs (for example an additional force) between the two masses is not permissible as we can only act on one of the masses. From these considerations it follows immediately that the maximum tolerable Δk for $k = k_0 = 1$ is given by $|\Delta k| < 1$. This implies that $\|\Delta(s)^1\|_\infty < 1$ and hence the optimally achievable \mathcal{H}_∞ -norm of the transfer function $T_\infty^1(C_0, s)$ from $w_\infty^1(s)$ to $z_\infty^1(s)$ is $\|T_\infty^1(C_0)\|_\infty = 1$, which in turn implies that \mathcal{H}_∞ -bounds $\gamma^1 < 1$ cannot be achieved. For the following controller design an \mathcal{H}_∞ -bound $\gamma^1 = 1.0005$ has been specified. The controller to be designed is a dynamic proper full-order controller with structural constraints defined by the controller system matrices below. Using the penalty function approach, a controller satisfying these requirements has been

designed. The (structurally constrained) controller realization is as follows.

$$A_c = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -4.96497865 & -4.81507032 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -8.34902647 & -3.16030032 \end{pmatrix},$$

$$B_c = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$C_c = \begin{pmatrix} 67.97641069 & 12.13736699 & 193.16591187 & 101.09971272 \end{pmatrix},$$

$$D_c = \begin{pmatrix} -69.72761009 & 32.89950444 \end{pmatrix}.$$

The singular value plot of the closed-loop transfer function $T_\infty^1(C_0, s)$ from $w_\infty^1(s)$ to $z_\infty^1(s)$ in Figure 5.3 verifies that the specified \mathcal{H}_∞ -bound is indeed satisfied. The designed controller guarantees stability for $k \in [k_0 - \frac{1}{1.0003}, k_0 + \frac{1}{1.0003}]$ and hence $k \in [3 \times 10^{-4}, 1.9997]$. This in turn implies that uncertainties Δk with $|\Delta k| < 0.9997$ can be tolerated without the closed-loop system becoming unstable. Some closed-loop properties are summarized in table 5.3.

5.5.2.2 Case 2: Measurement of $x_1^1(t)$ and Actuation on the First Mass

For this design case only the position of the first mass is available as measurement and a control action on the first mass is allowed. The open-loop system matrices B_3^1 and C_3^1 are as follows.

$$B_3^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad C_3^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}.$$

This design case has equivalent interpretations regarding the minimally achievable \mathcal{H}_∞ -norm of $\|T_\infty^1(C_0)\|_\infty$. That is, there is no controller that will satisfy a specified

Table 5.3: Two-mass spring \mathcal{H}_∞ -design example: Closed-loop properties for design case 1; Measurements of $x_1^1(t)$, $x_3^1(t)$; Actuation on the first mass.

Nominal design plant:	$m_0 = 1, k_0 = 1$
Specified γ^1 :	1.0005 (0.01 dB)
Achieved $\ T_\infty^1(C_0)\ _\infty$:	1.0003 (0.0060 dB)
Guaranteed stability region:	$3 \times 10^{-4} < k < 1.9997$
Closed-loop system poles:	$\lambda_1 = -0.0001$ $\lambda_2 = -0.1670$ $\lambda_3 = -1.3018$ $\lambda_4 = -3.3323$ $\lambda_{5,6} = -0.7674 \pm 8.4148j$ $\lambda_{7,8} = -0.8197 \pm 2.3065j$
Controller poles:	$\lambda_{c1} = -1.4958$ $\lambda_{c2} = -3.3193$ $\lambda_{c3,4} = -1.5802 \pm 2.4191j$

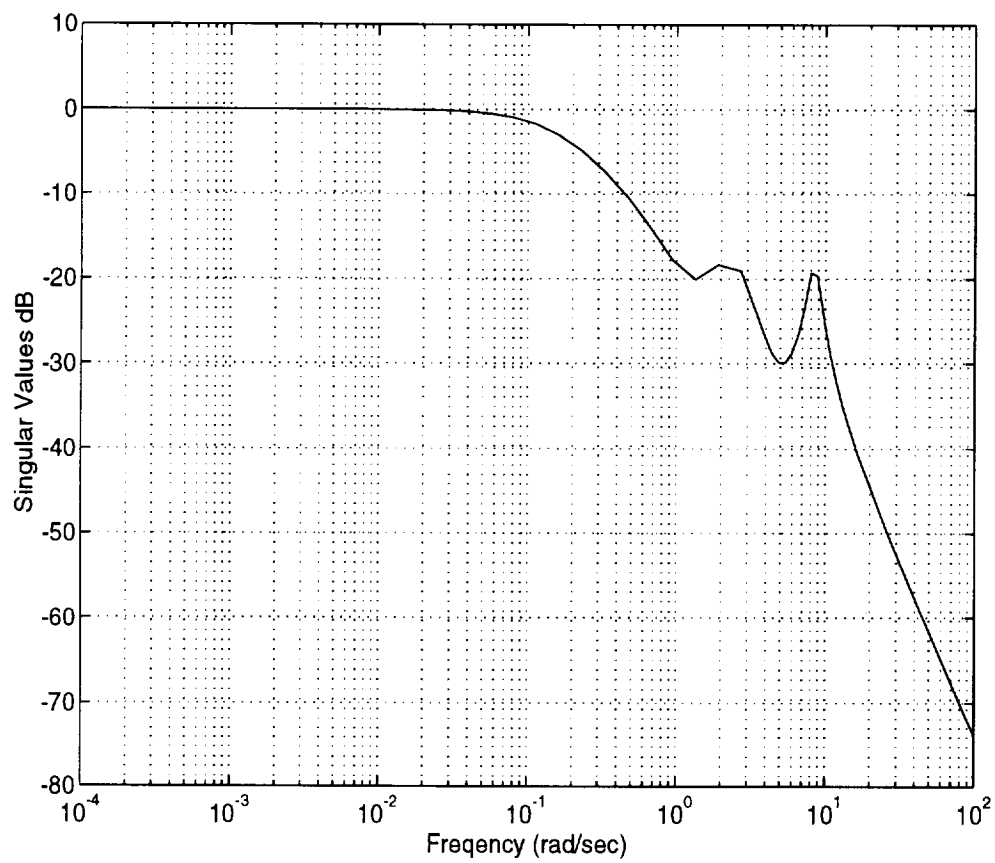


Figure 5.3: Two-mass spring \mathcal{H}_{∞} -design example: Singular value plot of $T_{\infty}^1(C_0, s)$ for the designed controller; Design case 1: Measurements of $x_1^1(t)$, $x_3^1(t)$; Actuation on the first mass.

\mathcal{H}_∞ -bound $\|T_\infty^1(C_0)\|_\infty < 1$. However, this design case differs from the first in that it represents a problem with a co-located sensor/actuator pair. Furthermore, as the position of the second mass is not available this variable has to be “reconstructed” in the controller to guarantee closed-loop stability. The specified \mathcal{H}_∞ -bound γ^1 for the design was set to $\gamma^1 = 1.05$. The controller structure was chosen such that controllability of the pair (A_c, B_c) is explicitly imposed (see the controller matrices below). The selected controller type is a dynamic proper full-order controller. The designed controller has a realization with the following state-space matrices.

$$A_c = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.04433820 & 5.66240636 & 0.44756456 & -4.32708050 \end{pmatrix}, \quad B_c = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$C_c = \begin{pmatrix} 0.40013604 & -51.10117437 & -3.95336090 & 27.45265681 \end{pmatrix},$$

$$D_c = \begin{pmatrix} -9.02463471 \end{pmatrix}$$

The closed-loop properties for this design are summarized in table 5.4. The singular value plot of the closed-loop system (see Figure 5.4) validates that the specified \mathcal{H}_∞ -bound $\gamma^1 = 1.05$ has been satisfied. The designed controller stabilizes the system and guarantees stability for $k \in [k_0 - \frac{1}{1.0415}, k_0 + \frac{1}{1.0415}]$ and hence $k \in [0.0398, 1.9602]$. This in turn implies that uncertainties Δk with $|\Delta k| < 0.9602$ can be tolerated without the system becoming unstable.

5.5.2.3 Case 3: Measurements of $x_1^1(t)$, $x_2^1(t)$ and Actuation on Both Masses

In the last design case for this example let us consider the problem where both positions are available as measurements and we can actuate on both masses. The open-loop system matrices B_3^1 and C_3^1 corresponding to this problem are as follows.

$$B_3^1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_3^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Table 5.4: Two-mass spring \mathcal{H}_∞ -design example: Closed-loop properties for design case 2; Measurement of $x_1^1(t)$; Actuation on the first mass.

Nominal design plant:	$m_0 = 1, k_0 = 1$
Specified γ^1 :	1.05 (0.9758 dB)
Achieved $\ T_\infty^1(C_0)\ _\infty$:	1.0415 (0.8132 dB)
Guaranteed stability region:	$0.0398 < k < 1.9602$
Closed-loop system poles:	$\lambda_1 = -0.0058$ $\lambda_2 = -0.0244$ $\lambda_{3,4} = -0.0025 \pm 0.0089j$ $\lambda_{5,6} = -0.8301 \pm 1.8971j$ $\lambda_{7,8} = -1.3158 \pm 0.1923j$
Controller poles:	$\lambda_{c1} = 0.0078$ $\lambda_{c2} = 1.0635$ $\lambda_{c3} = -1.2998$ $\lambda_{c4} = -4.0986$

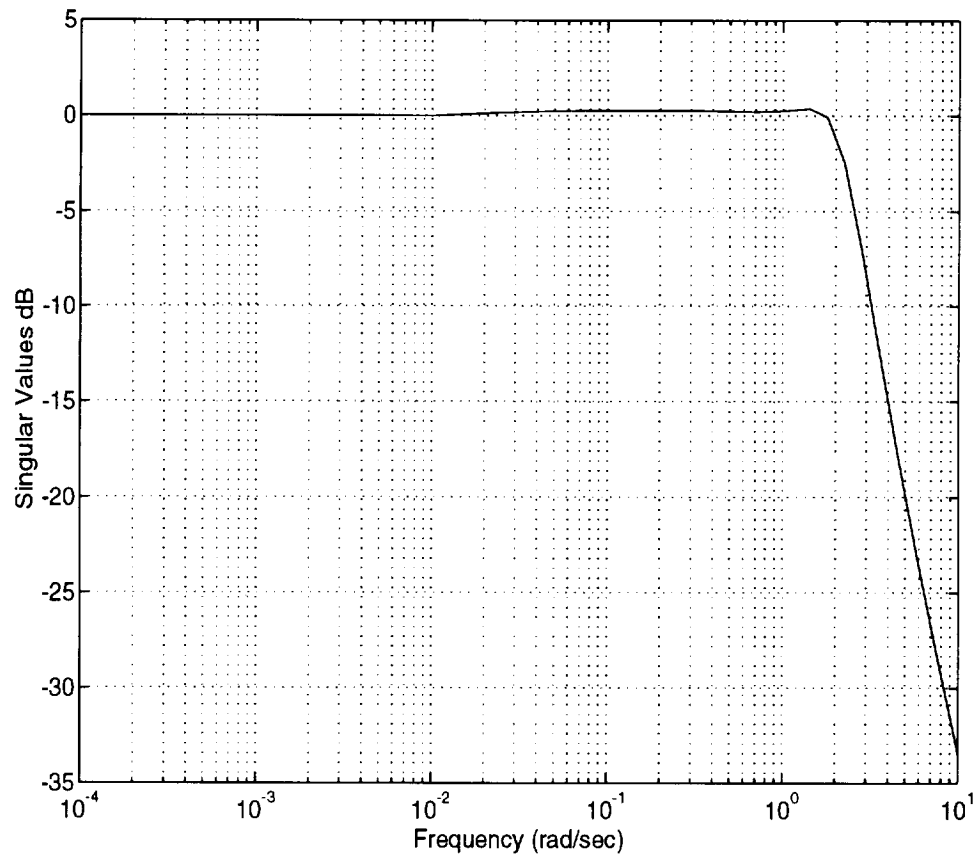


Figure 5.4: Two-mass spring \mathcal{H}_∞ -design example: Singular value plot of $T_\infty^1(C_0, s)$ for the designed controller; Design case 2: Measurement of $x_1^1(t)$; Actuation on the first mass.

This design case is interesting because the minimally achievable \mathcal{H}_∞ -norm of the transfer function $T_\infty^1(C_0, s)$ from $w_\infty^1(s)$ to $z_\infty^1(s)$ can be smaller than one. Obviously, if we can apply a force to both masses and both mass positions are available to the controller, then we can always compensate for any force in between these two masses (given sufficiently large control forces). This implies that even additional forces in between these two masses can be tolerated corresponding to the case of an “active” spring. Thus the achievable \mathcal{H}_∞ -norm for the transfer function $T_\infty^1(C_0, s)$ is $\|T_\infty^1(C_0)\|_\infty = 0$ and we can actually guarantee stability for any Δk . The controller type for this design case is a strictly proper fourth-order controller with structural constraints as depicted by the structure of the controller matrices A_c , B_c and C_c below. The resulting controller realization is given by the following state-space matrices.

$$A_c = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -447.13923701 & -57.63138721 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -35.90942974 & -127.57195209 \end{pmatrix},$$

$$B_c = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$C_c = \begin{pmatrix} 6170.60542737 & 10902.36023990 & -1038.25779087 & -9074.32543211 \\ -6181.52733530 & -10913.54138359 & 1038.72400509 & 9073.20212098 \end{pmatrix},$$

$$D_c = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The designed controller stabilizes the system and guarantees stability for $k \in [k_0 - \frac{1}{0.0888}, k_0 + \frac{1}{0.0888}]$ and hence for $k \in [-10.2613, 12.2613]$ (see table 5.5 and Figure 5.5). This implies that uncertainties Δk with $|\Delta k| < 11.2613$ can be tolerated without loss of stability. The lower limit in the bounding interval on k ($k \in [-10.2613, 12.2613]$) on the other hand is negative and indicates that a negative spring constants k within

the given boundaries will not cause instability. A negative spring constant, however, corresponds to an active element in between the two masses which can be interpreted as an additional force in addition to the passive spring between the two masses. This result confirms the above analysis on the minimally achievable \mathcal{H}_∞ -norm of $\|T_\infty^1(C_0)\|_\infty$. Note also that the first row in the controller matrix C_c is almost identical to the negated second row in C_c . This shows that both controls act in opposite directions so as to compensate for possible forces between the two masses. Furthermore, the more negative the acceptable k , the larger a possible force between these masses will be. Hence, for very small values of γ^1 this will generally require high-gain controllers and large control efforts.

Table 5.5: Two-mass spring \mathcal{H}_∞ -design example: Closed-loop properties for design case 3; Measurements of $x_1^1(t)$, $x_3^1(t)$; Actuation on both masses.

Nominal design plant:	$m_0 = 1, k_0 = 1$
Specified γ^1 :	0.1 (−20 dB)
Achieved $\ T_\infty^1(C_0)\ _\infty$:	0.0888 (−21.1103 dB)
Guaranteed stability region:	$-10.2613 < k < 12.2613$
Closed-loop system poles:	$\lambda_1 = -0.1026$ $\lambda_2 = -2.8778$ $\lambda_3 = -52.8723$ $\lambda_4 = -127.8507$ $\lambda_{5,6} = -0.0260 \pm 0.1256j$ $\lambda_{7,8} = -0.7238 \pm 16.1974j$
Controller poles:	$\lambda_{c1} = -0.2821$ $\lambda_{c2} = -9.2400$ $\lambda_{c3} = -48.3913$ $\lambda_{c4} = -127.2898$

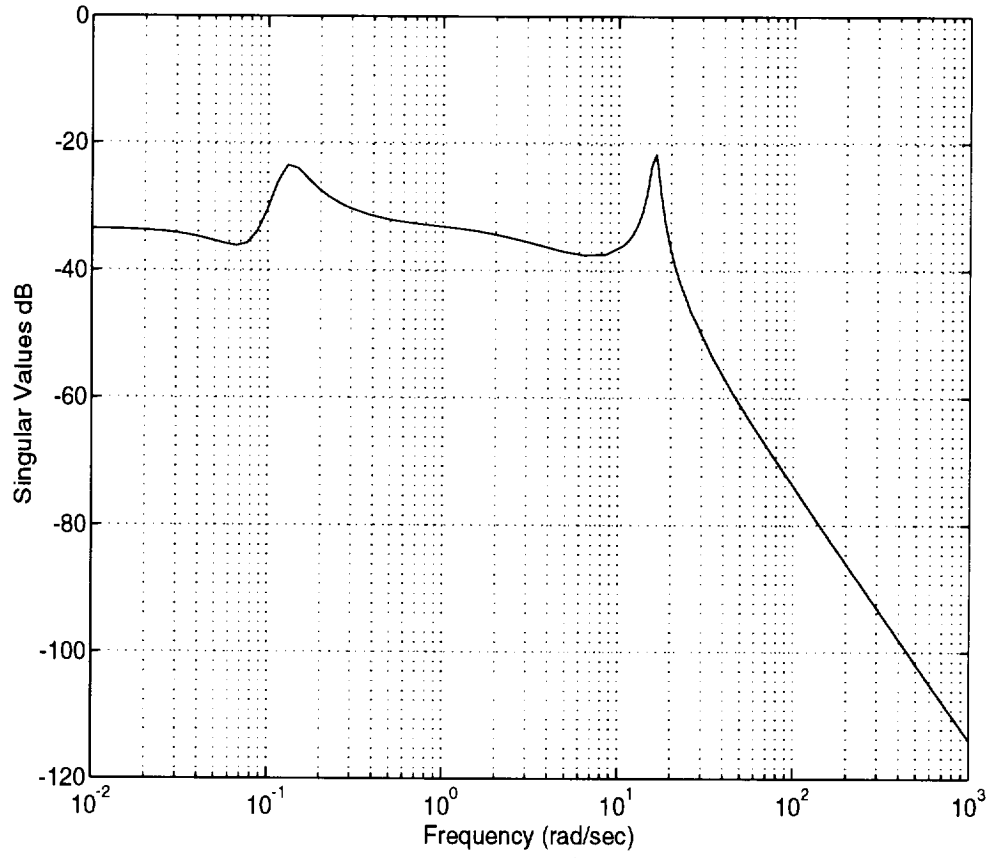


Figure 5.5: Two-mass spring \mathcal{H}_∞ -design example: Singular value plot of $T_\infty^1(C_0, s)$ for the designed controller; Design case 2: Measurements of $x_1^1(t)$, $x_3^1(t)$; Actuation on both masses.

5.5.3 Example 3: The Four-Disc Problem

The four-disc torsional system has been investigated thoroughly in the past decade (see e.g. [64] and references within). The system consists of four discs that are mounted on a vertical steel rod and can rotate around this rod. In the following description these discs are numbered according to their position in the system, i.e., the lowest disc is the first disc, the disc above that is disc number two and so forth. An actuation can be applied to this plant via a DC-motor attached to the third disc while measurements are available from sensors on the first disc. The inertia β of the uppermost disc (disc 4) is treated as uncertainty in this system. That is, it is assumed that $\beta = \beta_0 + \Delta\beta$ where the nominal value β_0 for the inertia of the first disc. The reader is referred to [64] for more details on the physical setup of this system. A corresponding state-space realization of this system with $\Delta\beta$ as uncertainty can be given as follows.

$$A^1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \beta_0 & 0 & -\beta_0 & 0 \end{pmatrix}, \quad B_2^1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad B_3^1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$C_2^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix},$$

$$C_3^1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where the 8 states $x_i^1(t)$, ($i = 1, 2, \dots, 8$) represent the angular positions $\theta_i(t)$ and angular rates $\Omega_i(t)$, $i = 1, 2, 3, 4$ of the four discs. That is, $x_1^1(t) = \theta_1(t)$, $x_2^1(t) = \Omega_1(t)$, $x_3^1(t) = \theta_2(t)$, $x_4^1(t) = \Omega_2(t)$, $x_5^1(t) = \theta_3(t)$, $x_6^1(t) = \Omega_3(t)$ and $x_7^1(t) = \theta_4(t)$, $x_8^1(t) = \Omega_4(t)$. The uncertainty block $\Delta^1(s)$ in this case is easily identified as $\Delta^1(s) = \Delta\beta$. The control input distribution matrix B_3^1 reveals the actuation on the third disc and C_3^1 indicates that the angular position and angular rate of the first disc are assumed to be measurable through sensors.

This plant has been considered in the framework of mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -control design with the inclusion of natural damping into the open-loop system (see e.g. [7]). With this modification the \mathcal{H}_∞ -design problem can be solved via the DGKF-approach as all the relevant system assumptions for this approach are satisfied. However, without any natural damping it can be verified that the \mathcal{H}_∞ -design problem for the above system cannot be solved by the DGKF-approach or the Stoorvogel solution to the suboptimal \mathcal{H}_∞ -design problem. Furthermore, without any natural damping this system exhibits some very interesting characteristics.

Let us assume that $\Delta\beta = 0$ and examine the pure stabilization problem of the nominal plant for some values of the inertia β_0 . It can be shown that the two nominal values $\beta_0 = 0.382$ and $\beta_0 = 2.618$ represent two special plant conditions. At $\beta_0 = 0.382$ and $\beta_0 = 2.618$ the system exhibits pole-zero cancellations on the $j\omega$ -axis. For these inertia values the resulting undamped modes are not controllable through B_3^1 and hence are not stabilizable. This implies that a controller C_0 can stabilize the above system only in the following intervals for β_0 :

$$\begin{aligned} 0 &< \beta_0 < 0.382 \\ 0.382 &< \beta_0 < 2.618 \\ 2.618 &< \beta_0. \end{aligned}$$

However, no controller exists that can stabilize the system for β_0 -intervals that contain either $\beta_0 = 0.382$ or $\beta_0 = 2.618$.

Now let us return to the problem of robustly stabilizing the above plant for $\Delta\beta \neq 0$ and $\beta = \beta_0 + \Delta\beta$. The nominal point β_0 of the physical system is $\beta_0 = 1$. Hence, if one is to design a controller at this nominal point, the maximally tolerable uncertainty $\Delta\beta$ can be computed to be

$$\begin{aligned} \Delta\beta_{max} &= \min[(\beta_0 - 0.382), (2.618 - \beta_0)] \\ &= \min[0.6180, 1.618] \\ &= 0.6180. \end{aligned}$$

This implies that the minimally achievable \mathcal{H}_∞ -norm $\|T_\infty^1(C_0)\|_\infty$ is $\|T_\infty^1(C_0)\|_\infty = \frac{1}{0.6180} = 1.6181$. However, for the controller synthesis one is not required to use the

physical nominal point. In order to maximize the region of robust stability one can select an alternate nominal point for the design as long as the resulting controller stabilizes the physically nominal plant as well. This has been done for the first design case in the next subsection. There a nominal value $\beta_0 = 1.5$ has been selected and a suboptimal \mathcal{H}_∞ -design problem for the single-plant case is performed.

The second design case assumes two plant conditions, one at $\beta_0 = 1$ and the second one at $\beta_0 = 4$. For these two conditions a single controller is designed that stabilizes both systems and satisfies the specified \mathcal{H}_∞ -bounds.

5.5.3.1 Case 1: Single-Plant Design

The nominal (design) point $\beta_0^1 = 1.5$ for this design case is approximately half way in between the theoretical limits $0.382 < \beta < 2.618$ for which a controller can stabilize the plant. Robust stability in the regions $0 < \beta < 0.382$ and $2.618 < \beta$ is not explicitly taken into account. It can be verified that with $\beta_0 = 1.5$ the theoretically smallest achievable \mathcal{H}_∞ -norm $\|T_\infty^1(C_0)\|_\infty$ is $\|T_\infty^1(C_0)\|_\infty = 0.8945$. For the following design the specified γ^1 has been set to $\gamma^1 = 1.2 > 0.8945$. The controller type is a fixed-order (fourth-order) proper controller that is structurally constrained. The selected controller structure realizes a PID-type control law. The only measurement available to the dynamic (PI) part of the controller is the angular position of the first disc. The differential portion is realized by the entry $D_{c,(1,2)}$ in the direct feedthrough term D_c of the proper controller. This term represents a proportional control law for the angular rate of the first disc and hence a differential controller for the position of the first disc. The designed controller has the following state-space realization:

$$A_c = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -15.8962925007 & -23.6235424835 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3.9218133071 & -0.8615188726 \end{pmatrix},$$

$$B_c = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$C_c = \begin{pmatrix} 31.5880796759 & -2.3698196389 & 13.3336407399 & 3.3435177407 \end{pmatrix},$$

$$D_c = \begin{pmatrix} -5.3906473307 & 2.4316125330 \end{pmatrix}$$

The controller stabilizes the nominal (design) system with $\beta_0^1 = 1.5$ (see table 5.6). The achieved \mathcal{H}_∞ -norm guarantees stability of the closed-loop system for any $\beta = \beta_0^1 + \Delta\beta$ for $0.5811 \leq \beta \leq 2.4189$ and hence also stabilizes the nominal point of the physical system ($\beta_0 = 1$). The controller results in a closed-loop system that is very close to being unstable (note that one of the closed-loop eigenvalues is $-1.9580 \times 10^{-6} \pm 1.6180j$). However, no eigenvalue constraints have been included in this design and hence the specified criterion – namely the \mathcal{H}_∞ -constraint – is satisfied. The singular value plot of $T_\infty^1(C_0, s)$ in Figure 5.6 confirms this fact. Unfortunately the designed controller does not stabilize the plant for values $\beta = \beta_0^1 + \Delta\beta$ other than $0.5811 \leq \beta \leq 2.4189$. If one requires stability in one (or both) of the above intervals as well, one has to formulate the \mathcal{H}_∞ -robust stability problem in terms of a multi-plant design problem. This will be done in the next section.

Table 5.6: Four-disc \mathcal{H}_∞ -design example: Closed-loop properties for design case 1: Single-plant case, $\beta_0^1 = 1.5$.

Nominal design point:	$\beta_0^1 = 1.5$
Specified γ^1 :	1.2 (1.5836 dB)
Achieved $\ T_\infty^1(C_0)\ _\infty$:	1.0883 (0.7350 dB)
Guaranteed stability region:	$0.5811 \leq \beta \leq 2.4189$
Closed-loop system poles:	$\lambda_1 = -0.0065$ $\lambda_2 = -22.9303$ $\lambda_{3,4} = -1.9580 \times 10^{-6} \pm 1.6180j$ $\lambda_{5,6} = -0.0347 \pm 0.3266j$ $\lambda_{7,8} = -0.1387 \pm 2.0247j$ $\lambda_{9,10} = -0.2913 \pm 0.8208j$ $\lambda_{11,12} = -0.3093 \pm 1.5853j$
Controller poles:	$\lambda_{c1} = -0.6932$ $\lambda_{c2} = -22.9302$ $\lambda_{c3,4} = -0.4307 + 1.9329j$

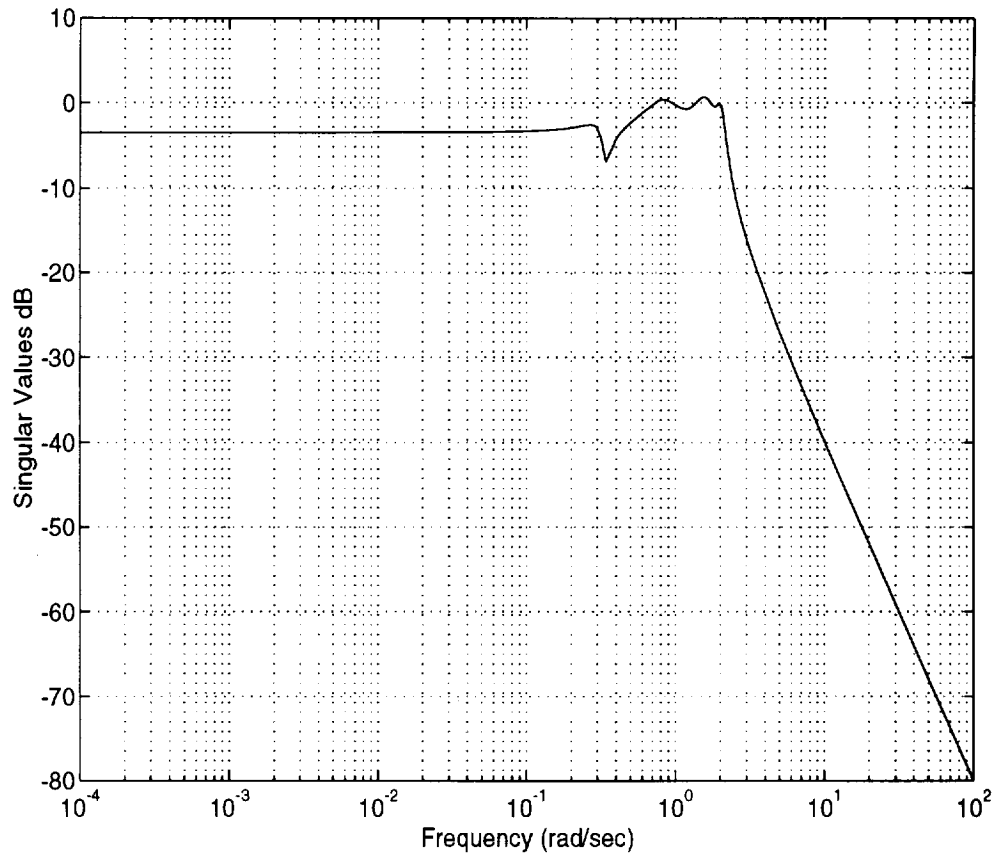


Figure 5.6: Four-disc \mathcal{H}_∞ -design example: Singular value plot of $T_\infty^1(C_0, s)$ for design case 1: Single-plant case, $\beta_0^1 = 1.5$.

5.5.3.2 Case 2: Multi-Plant Design

Here the nominal points for the two (design) systems have been chosen to be $\beta_0^1 = 1$ and $\beta_0^2 = 4$. By setting $\beta_0^1 = 1$ the first plant condition represents the physically nominal system. With this choice a single controller is sought that simultaneously stabilizes both plant conditions and provides robust stability for some regions in the interval $0.382 < \beta < 2.618$ (around the physical nominal point $\beta_0^1 = 1$) and in the interval $2.618 < \beta$ (around the nominal point $\beta_0^2 = 4$). Note that β here stands for the inertia of the real plant, the values β_0^1 and β_0^2 are only selected design parameters. The desired stability regions depend on the specified \mathcal{H}_∞ -bounds γ^1 and γ^2 . For this multi-plant example these values have been chosen to be $\gamma^1 = \gamma^2 = 2$ and do not violate the theoretical limits for the achievable \mathcal{H}_∞ -norms which are $\|T_\infty^1(C_0)\|_\infty = 1.6181$ and $\|T_\infty^2(C_0)\|_\infty = 0.7236$ respectively. The controller type and structure are the same as in design case 1 for this example. The designed controller has the following state-space realization:

$$A_c = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -4.4140039143 & -6.3059498695 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -5.442805186824 & -0.6079752614 \end{pmatrix},$$

$$B_c = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$C_c = \begin{pmatrix} 152.8606825147 & 183.2410250521 & 40.4698252837 & 1.2192732255 \end{pmatrix},$$

$$D_c = \begin{pmatrix} -42.1208298559 & 8.2785456342 \end{pmatrix}.$$

The closed-loop properties for both plant conditions show that the controller stabilizes both plant conditions simultaneously and satisfies the specified \mathcal{H}_∞ -bounds (see tables 5.7 and 5.8). From the achieved \mathcal{H}_∞ -norms $\|T_\infty^1(C_0)\|_\infty$ and $\|T_\infty^2(C_0)\|_\infty$ the achieved stability regions in terms of β can be computed. With this controller, closed-loop stability is guaranteed for all β in the intervals $0.4981 \leq \beta \leq 1.5019$,

$3.4384 \leq \beta \leq 4.5616$. The singular value plots of both closed-loop plant conditions are shown in Figures 5.7 and 5.8 respectively. However, the closed-loop system remains unstable for any value of β in $0 < \beta < 0.382$. If stability in this region is required in addition, one has to define a \mathcal{H}_∞ -design problem with three plant conditions to explicitly account for this specification.

Table 5.7: Four-disc \mathcal{H}_∞ -design example: Closed-loop properties of the first plant condition for design case 2: Multi-plant case, $\beta_0^1 = 1$.

Nominal design point:	$\beta_0^1 = 1$
Specified γ^1 :	2 (6.0206 dB)
Achieved $\ T_\infty^1(C_0)\ _\infty$:	1.9925 (5.9880 dB)
Guaranteed stability region:	$0.4981 \leq \beta \leq 1.5019$
Closed-loop system poles:	$\lambda_1 = -0.3870$ $\lambda_2 = -5.4984$ $\lambda_{3,4} = -0.0003 \pm 1.6179j$ $\lambda_{5,6} = -0.0685 \pm 0.1881j$ $\lambda_{7,8} = -0.1032 \pm 0.6712j$ $\lambda_{9,10} = -0.1575 \pm 1.4693j$ $\lambda_{11,12} = -0.1847 \pm 2.4067j$
Controller poles:	$\lambda_{c1} = -0.8020$ $\lambda_{c2} = -5.5040$ $\lambda_{c3,4} = -0.3040 \pm 2.3131j$

Table 5.8: Four-disc \mathcal{H}_∞ -design example: Closed-loop properties of the second plant condition for design case 2: Multi-plant case, $\beta_0^2 = 4$.

Nominal design point:	$\beta_0^2 = 4$
Specified γ^2 :	2 (6.0206 dB)
Achieved $\ T_\infty^2(C_0)\ _\infty$:	1.7805 (5.0108 dB)
Guaranteed stability region:	$3.4384 \leq \beta \leq 4.5616$
Closed-loop system poles:	$\lambda_1 = -0.2394$ $\lambda_2 = -5.4984$ $\lambda_{3,4} = -0.0001 \pm 1.6181j$ $\lambda_{5,6} = -0.0619 \pm 0.2784j$ $\lambda_{7,8} = -0.0903 \pm 2.4435j$ $\lambda_{9,10} = -0.1568 \pm 2.1942j$ $\lambda_{11,12} = -0.2791 \pm 0.7536j$
Controller poles:	$\lambda_{c1} = -0.8020$ $\lambda_{c2} = -5.5040$ $\lambda_{c3,4} = -0.3040 \pm 2.3131j$

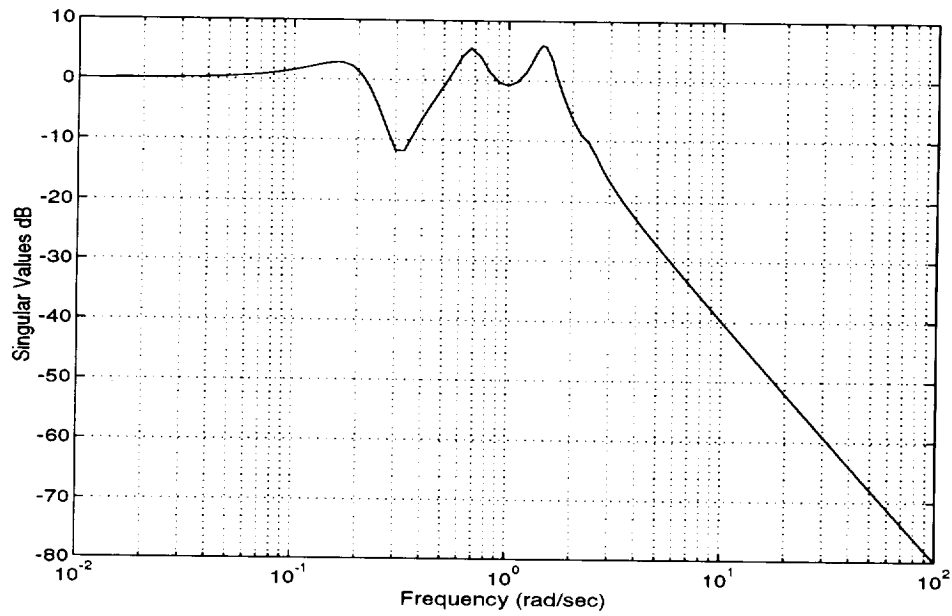


Figure 5.7: Four-disc \mathcal{H}_∞ -design example: Singular value plot of $T_\infty^1(C_0, s)$ for design case 2: Multi-plant case, $\beta_0^1 = 1$.

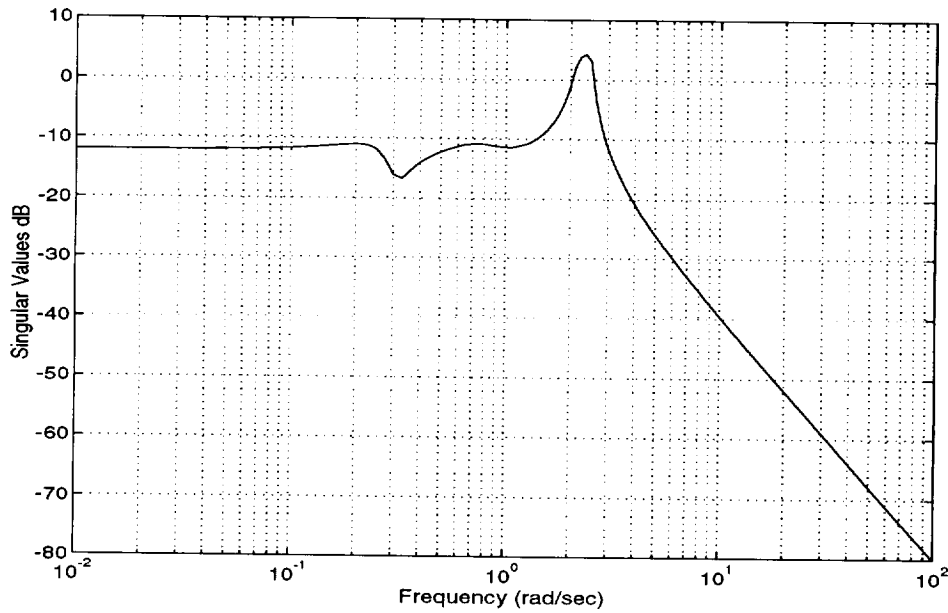


Figure 5.8: Four-disc \mathcal{H}_∞ -design example: Singular value plot of $T_\infty^2(C_0, s)$ for design case 2: Multi-plant case, $\beta_0^2 = 4$.

Chapter 6

THE MIXED $\mathcal{H}_2/\mathcal{H}_\infty$ -PROBLEM

6.1 The General Multi-Plant Case

The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design problem stated in definition 3.2.1 of chapter 3 is a constrained optimization problem. With n_p plant conditions $\Sigma_{2,cl}^i(C_0)$ and $\Sigma_{\infty,cl}^i(C_0)$ and their respective transfer functions $T_2^i(C_0, s)$ and $T_\infty^i(C_0, s)$ defined in chapter 3, a set of n_p prespecified weighting parameters α^i and the set of fixed robust stability bounds γ^i , an internally stabilizing controller C_0 is to be found that solves the following optimization problem.

$$\min_{C_0} \lim_{t_{f\mathcal{H}_2} \rightarrow \infty} J_2(C_0, t_{f\mathcal{H}_2}) = \min_{C_0} \lim_{t_{f\mathcal{H}_2} \rightarrow \infty} \sum_{i=1}^{n_p} \alpha^i J_2^i(C_0, t_{f\mathcal{H}_2}), \quad (6.1)$$

where

$$J_2^i(C_0, t_{f\mathcal{H}_2}) = \mathcal{E}[z_2^{iT}(t_{f\mathcal{H}_2}) z_2^i(t_{f\mathcal{H}_2})] \quad (6.2)$$

$$\lim_{t_{f\mathcal{H}_2} \rightarrow \infty} J_2^i(C_0, t_{f\mathcal{H}_2}) = \|T_2^i(C_0)\|_2^2, \quad (6.3)$$

subject to n_p \mathcal{H}_∞ -robust stability constraints

$$\|T_\infty^i(C_0)\|_\infty < \gamma^i, \quad i = 1, 2, \dots, n_p. \quad (6.4)$$

The pure multi-plant \mathcal{H}_∞ -design problem has been discussed thoroughly in chapter 5. The trace-type penalty cost function defined in chapter 4 and the penalty/barrier function approach for the pure \mathcal{H}_∞ -design problem immediately lead to the following formulation of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design problem. Namely, the n_p \mathcal{H}_∞ -constraints are formulated in terms of the tools developed in the chapter 5. Following this approach the n_p \mathcal{H}_∞ -constraints are transformed into a set of matrix inequalities which in turn are represented in terms of scalar cost functions. Ultimately the robust stability problem for all n_p plant conditions is expressed in terms of the overall cost function $J_\infty(C_0, \mathcal{X}, \mathcal{G}, T_f)$ defined in chapter 5. This cost function is zero in the limit

as $\mathcal{T}_f \rightarrow \infty$, if and only if all the matrix constraints and hence all \mathcal{H}_∞ -constraints imposed on the n_p closed-loop systems $\Sigma_{\infty,cl}^i(C_0)$ are satisfied. For the following development it is assumed that the sets \mathcal{X} , \mathcal{G} and \mathcal{T}_f are the same as defined in chapter 5.

As with the pure \mathcal{H}_∞ -problem, there are various possible ways to pose this optimization problem in a gradient-based framework. Numerically the problem at hand can be formulated as a constrained optimization problem. Utilizing the cost function $J_\infty(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f)$, one can define a constrained minimization problem to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -problem as follows.

$$\min_{C_0, \mathcal{X}} \lim_{t_{f\mathcal{H}_2} \rightarrow \infty} J_2(C_0, t_{f\mathcal{H}_2}), \quad (6.5)$$

subject to the single constraint

$$\lim_{\mathcal{T}_f \rightarrow \infty} J_\infty(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f) = 0, \quad (6.6)$$

where, alternatively, the single constraint (6.6) can be expressed as a set of $3n_p$ constraints such as in (5.62)–(5.64).

If the closed-loop systems are allowed to become unstable during intermediate phases of the optimization, then the corresponding \mathcal{H}_2 -norms $\|T_2^i(C_0)\|_2^2$ are not defined and one has to use the cost function $J_2^i(C_0, t_{f\mathcal{H}_2})$ for a finite $t_{f\mathcal{H}_2}$ and solve the above optimization problem for a monotonically increasing $t_{f\mathcal{H}_2}$. Once the controller is stabilizing all the plant conditions, then in the limit, as $t_{f\mathcal{H}_2} \rightarrow \infty$, $J_2^i(C_0, t_{f\mathcal{H}_2})$ recovers the exact performance measure that we want to minimize (see (3.22)). This scheme has been developed in [64] and the formulation for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design problem as worked out in this research fits nicely into the established \mathcal{H}_2 -design framework.

On the other hand, if the controller is restricted to stabilize all plant conditions during the whole course of the optimization, then Lyapunov equation solutions can be utilized to compute the exact function values $\|T_2^i(C_0)\|_2^2$ and the corresponding gradients for all plant conditions (see (3.24)–(3.27)). However, such an optimization scheme must assure that the controller remains stabilizing during every phase of the optimization.

Following the spirit in chapter 5, the multi-plant mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design problem is formulated via a penalty/barrier function represented by $J_\infty(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f)$.

Definition 6.1.1

Under the assumptions in definition 3.2.1 an unconstrained mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -cost function $J_{2/\infty}(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f, t_{f\mathcal{H}_2},)$ is defined as follows.

$$J_{2/\infty}(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f, t_{f\mathcal{H}_2}) = c_2 J_2(C_0, t_{f\mathcal{H}_2}) + J_\infty(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f) \quad (6.7)$$

with $J_2(C_0, t_{f\mathcal{H}_2})$ given in definition 3.2.1, $J_\infty(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f)$ as in chapter 5 and a scaling factor $c_2 > 0$. In the limit, as $\mathcal{T}_f \rightarrow \infty$ and $t_{f\mathcal{H}_2} \rightarrow \infty$, the optimization problem

$$J_{2/\infty}^*(C_0^*, \mathcal{X}^*, \mathcal{G}) = \min_{C_0, \mathcal{X}} \lim_{\substack{\mathcal{T}_f \rightarrow \infty \\ t_{f\mathcal{H}_2} \rightarrow \infty}} J_{2/\infty}(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f, t_{f\mathcal{H}_2}) \quad (6.8)$$

solves the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design strategy in definition 3.2.1 provided there exists a controller that satisfies all the \mathcal{H}_∞ -bounds.

If a controller C_0^* and a corresponding set \mathcal{X}^* have been found such that all the n_p \mathcal{H}_∞ -constraints are satisfied and all the relevant ARI-matrix constraints in definition 5.1.1 are satisfied such that $ARI_{C,OF}^i(C_0^*, (X^i)^*, \gamma^i) < 0$, $R^i > 0$ and $(X^i)^* > 0$ for $i = 1, 2, \dots, n_p$, then

$$\lim_{\mathcal{T}_f \rightarrow \infty} J_\infty(C_0^*, \mathcal{X}^*, \mathcal{G}, \mathcal{T}_f) = 0.$$

Note in particular, that a controller C_0^* that satisfies this condition automatically stabilizes all plant conditions as discussed in chapter 3. This implies that stability of all closed-loop matrices A_{cl}^i is a natural result of this process and needs not be enforced as an additional constraint in this formulation. This is important since establishing stability constraints either requires additional optimization variables or, if defined in terms of direct constraints on their eigenvalues, these constraints may not always be differentiable. Hence, for a controller C_0^* that satisfies all \mathcal{H}_∞ -constraints and hence stabilizes all the plant conditions, the n_p \mathcal{H}_2 -norms $\|T_2^i(C_0^*)\|_2^2$ are well defined and we have, along with a corresponding set \mathcal{X}^* and (6.6), that

$$\lim_{\substack{\mathcal{T}_f \rightarrow \infty \\ t_{f\mathcal{H}_2} \rightarrow \infty}} J_{2/\infty}(C_0^*, \mathcal{X}^*, \mathcal{G}, \mathcal{T}_f, t_{f\mathcal{H}_2}) < \infty.$$

Moreover, in this case,

$$\lim_{\substack{\mathcal{T}_f \rightarrow \infty \\ t_{f\mathcal{H}_2} \rightarrow \infty}} J_{2/\infty}(C_0^*, \mathcal{X}^*, \mathcal{G}, \mathcal{T}_f, t_{f\mathcal{H}_2}) = \sum_{i=1}^{n_p} \alpha^i \|T_2^i(C_0^*)\|_2^2, \quad (6.9)$$

which is the \mathcal{H}_2 -performance cost that needs to be minimized. On the other hand, if one or more of the \mathcal{H}_∞ -constraint are not satisfied or the controller C_0^* is not internally stabilizing one or more of the closed-loop plants, then either $J_2(C_0^*, t_{f\mathcal{H}_2})$ or $J_\infty(C_0^*, \mathcal{X}^*, \mathcal{G}, T_f)$ or both will be unbounded in the limit as $T_f \rightarrow \infty$ and $t_{f\mathcal{H}_2} \rightarrow \infty$ and hence the overall cost function will be unbounded in this case as well. Thus in this limit, $J_{2/\infty}(C_0^*, \mathcal{X}^*, \mathcal{G}, T_f, t_{f\mathcal{H}_2})$ is finite if and only if C_0^* stabilizes all plant conditions and the set \mathcal{X}^* is such that all n_p ARI-constraints are satisfied. In the following the expressions $J_{2/\infty}(C_0^*, \mathcal{X}^*, \mathcal{G}, T_f, t_{f\mathcal{H}_2} \rightarrow \infty)$ and $J_2(C_0^*, t_{f\mathcal{H}_2} \rightarrow \infty)$ are abbreviated by $J_{2/\infty}(C_0^*, \mathcal{X}^*, \mathcal{G}, T_f, \infty)$ and $J_2(C_0^*, \infty)$ respectively. This is a nice property of the defined cost function for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design. It also suggests an iterative procedure similar to the one proposed in the last chapter to numerically solve the design problem. In [64] explicit gradient expressions for finite-time cost functions such as $J_2(C_0^*, t_{f\mathcal{H}_2})$ have been derived and thus the problem can be solved for an increasing sequence of $t_{f\mathcal{H}_2}$ and increasing elements in T_f as a sequence of unconstrained minimization problems. Alternatively, in this research a barrier function approach has been applied to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design problem. The outline of such an algorithm is as follows.

1. Initialization:

Specify a set \mathcal{G} of desired \mathcal{H}_∞ -bounds and a set of weighting factors α^i , $i = 1, 2, \dots, n_p$. Select an initial controller guess C_0 of the desired structure and order and an initial guess for the set \mathcal{X} as described in the last chapter.

2. Phase one: Computation of a \mathcal{H}_∞ -controller:

Using the machinery developed in chapter 5, find a controller C_0^0 that satisfies all n_p \mathcal{H}_∞ -constraints and stabilizes all plant conditions as well as a set \mathcal{X}^0 such that all ARI-related matrix constraints in definition 5.1.1 and

$$\lim_{T_f \rightarrow \infty} J_\infty(C_0^0, \mathcal{X}^0, \mathcal{G}, T_f) = 0$$

are satisfied. If no such controller can be found, the algorithm terminates here.

3. Phase two: Computation of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -controller:

Set $t_{f\mathcal{H}_2} = \infty$ and select an initial $(\mathcal{T}_f)^1$ and the scaling factor $(c_2)^1$ such that

$$(c_2)^1 J_2(C_0^0, \infty) = (c_2)^1 \sum_{i=1}^{n_p} \alpha^i \|T_2^i(C_0^0)\|_2^2 = 1 \quad (6.10)$$

$$J_\infty(C_0^0, \mathcal{X}^0, \mathcal{G}, (\mathcal{T}_f)^1) \ll 1. \quad (6.11)$$

With these settings and using C_0^0 and \mathcal{X}^0 as initial guesses, set $k = 1$ and perform the following steps at the k^{th} iteration:

- Solve the unconstrained minimization problem

$$\min_{C_0, \mathcal{X}} J_{2/\infty}(C_0, \mathcal{X}, \mathcal{G}, (\mathcal{T}_f)^k, \infty) \quad (6.12)$$

to get C_0^k and $(\mathcal{X})^k$.

- If

$$\left| \sum_{i=1}^{n_p} \alpha^i \|T_2^i(C_0^{k-1})\|_2^2 - \sum_{i=1}^{n_p} \alpha^i \|T_2^i(C_0^k)\|_2^2 \right| < \varepsilon \quad (6.13)$$

for some prespecified ε , then stop. Otherwise increase the elements in $(\mathcal{T}_f)^k$ to form $(\mathcal{T}_f)^{k+1}$ and increase $(c_2)^k$ to $(c_2)^{k+1}$ such that

$$(c_2)^{k+1} \sum_{i=1}^{n_p} \alpha^i \|T_2^i(C_0^k)\|_2^2 = 1 \quad (6.14)$$

and

$$J_\infty(C_0^k, \mathcal{X}^k, \mathcal{G}, (\mathcal{T}_f)^{k+1}) \ll 1 \quad (6.15)$$

and repeat the minimization (6.12).

This bootstrap method of first computing a controller that satisfies all the \mathcal{H}_∞ -constraints before addressing the mixed performance/robustness strategy has many advantages. In general the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -problem has a solution if and only if the pure \mathcal{H}_∞ -problem is solvable, that is, if there is a controller that satisfies all n_p specified \mathcal{H}_∞ -constraints. Without the existence of such a controller the mixed problem has no solution. Hence, if in phase one of the above algorithm no controller can be found that satisfies all the \mathcal{H}_∞ -constraints, then there is no need to initiate phase two and the algorithm terminates at this point.

However, because the matrix inequalities representing the \mathcal{H}_∞ -constraints and hence the overall \mathcal{H}_∞ -design algorithm are in general not convex, such a negative outcome does not necessarily imply that there is no controller at all that satisfies the desired \mathcal{H}_∞ -constraints. Hence, as with every optimization problem, the initial guesses and physical insights into the problem are of importance. However, if a controller has been found that does satisfy the \mathcal{H}_∞ -constraints, then the non-uniqueness of this whole class of controllers can be exploited to additionally minimize the performance cost as it is done in phase two of the algorithm. The initial guess for the overall algorithm is not required to be stabilizing all the plant conditions; in general it can be arbitrary. The controller C_0^0 derived in phase one of the algorithm not only satisfies all the \mathcal{H}_∞ -bounds but will also stabilize all n_p plant conditions. In the second phase of the algorithm the above updates for $(T_f)^k$ and $(c_2)^k$ guarantee that all \mathcal{H}_∞ -related constraints, i.e. $ARI_{C_0,OF}^i(C_0, X^i, \gamma^i) < 0$, $D_{cl,\infty}^{iT} D_{cl,\infty}^i - (\gamma^i)^2 I < 0$ and $X^i > 0$ for $i = 1, 2, \dots, n_p$ remain in effect throughout the whole optimization. Hence $J_\infty(C_0^k, \mathcal{X}^k, \mathcal{G}, (T_f)^{k+1})$ acts as a barrier function in the $(k+1)^{th}$ iteration, rejecting controllers that violate any of the n_p \mathcal{H}_∞ -constraints. This also implies that in this phase the search will be performed only over the set of stabilizing controllers. Hence we can use $J_2(C_0, t_{f\mathcal{H}_2})$ with the limit $t_{f\mathcal{H}_2} \rightarrow \infty$. In this case $J_2(C_0, \infty) = \sum_{i=1}^{n_p} \alpha^i \|T_2^i(C_0)\|_2^2$ represents the performance cost in the second phase of the optimization. In this framework, the \mathcal{H}_2 -performance cost and the respective gradients can be computed via Lyapunov solutions and not as a finite-time cost function ([64]).

The computation of this cost in terms of the grammians has been described in chapter 3 and the corresponding gradients for $\|T_2^i(C_0)\|_2^2$ have been derived in appendix B. The overall gradient expressions of $J_{2/\infty}(C_0, \mathcal{X}, \mathcal{G}, T_f, \infty)$ with respect to X^i are identical to the respective partial gradients of $J_\infty(C_0, \mathcal{X}, \mathcal{G}, T_f)$ as the \mathcal{H}_2 -performance cost is not a function of \mathcal{X} . The gradients of the overall cost function with respect to C_0 are as follows.

$$\begin{aligned} \frac{\partial J_{2/\infty}(C_0, \mathcal{X}, \mathcal{G}, T_f, \infty)}{\partial C_0} &= \sum_{i=1}^{n_p} \{ \bar{C}_3^i L_1^i [L_2^i \bar{B}_3^i + (C_{cl,2}^i)^T \bar{D}_{23}^i] \}^T \\ &\quad + \frac{\partial J_\infty(C_0, \mathcal{X}, \mathcal{G}, T_f)}{\partial C_0}, \end{aligned} \quad (6.16)$$

where L_1^i and L_2^i solve

$$A_{cl}^i L_1^i + L_1^i (A_{cl}^i)^T + B_{cl,2}^i (B_{cl,2}^i)^T = 0 \quad (6.17)$$

$$L_2^i A_{cl}^i + (A_{cl}^i)^T L_2^i + (C_{cl,2}^i)^T C_{cl,2}^i = 0 \quad (6.18)$$

respectively. With $J_\infty(C_0, \mathcal{X}, \mathcal{G}, \mathcal{T}_f)$ defined in (5.65), no further modifications are needed to form a well-conditioned optimization problem. That is, the additional performance cost does not introduce any unforeseen difficulty in terms of new undesirable local minima.

Taking into account that the algorithm will terminate at some $(\mathcal{T}_f)^k$ with finite elements, it is obvious that $J_\infty(C_0^k, \mathcal{X}^k, \mathcal{G}, (\mathcal{T}_f)^{k+1})$ will never be exactly zero and hence the performance cost $(c_2)^{k+1} \sum_{i=1}^{n_p} \alpha^i \|T_2^i(C_0^k)\|_2^2$ will always be larger than the optimally achievable. The optimally achievable performance cost will be achieved only in the limit as $(\mathcal{T}_f)^k \rightarrow \infty$. However, since in the second phase of the algorithm all the \mathcal{H}_∞ -constraints will remain satisfied, there will be n_p small but positive ε such that $ARI_{C,OF}^i(C_0, X^i, \gamma^i) + \varepsilon I < 0$, $D_{cl,\infty}^{iT} D_{cl,\infty}^i - (\gamma^i)^2 I + \varepsilon I < 0$ and $-X^i + \varepsilon I < 0$. Hence there will be large but finite t_{f1}^i , t_{f2}^i and t_{f3}^i for each plant condition such that $J_\infty(C_0^k, \mathcal{X}^k, \mathcal{G}, (\mathcal{T}_f)^{k+1})$ can be made arbitrarily small and hence we can approach the optimally achievable performance as close as desired using the above finite-time algorithm. This fact is reflected in the termination criterion for the second phase, namely $|\sum_{i=1}^{n_p} \alpha^i \|T_2^i(C_0^{k-1})\|_2^2 - \sum_{i=1}^{n_p} \alpha^i \|T_2^i(C_0^k)\|_2^2| < \varepsilon$.

As discussed before, there are principally two different design problems. For the simultaneous mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design problem one seeks to find a controller that is \mathcal{H}_2 -optimal, i.e. a controller that minimizes the above \mathcal{H}_2 -cost and additionally satisfies the considered \mathcal{H}_∞ -bounds. Note that this problem may not be solvable even if a controller exists that satisfies all the specified \mathcal{H}_∞ -constraints. If such a controller exists, however, it can be expected that the achieved \mathcal{H}_∞ -norms are not on the specified boundary. That is, there is no competition between the performance and the robust stability objectives. The general mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design problem addresses cases where the two objectives compete and a controller that satisfies the desired \mathcal{H}_∞ -bounds will not be \mathcal{H}_2 -optimal. In this case the achieved \mathcal{H}_∞ -norms will generally be at the specified boundary. The class of simultaneous mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -controllers is generally a subclass of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -controllers. The presented formulation is applicable to both problems. However, due to the non-convexity of the overall prob-

lem it cannot be guaranteed that the above algorithm will converge to a simultaneous mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -controller – if such a controller exists for the problem at hand.

If a controller of the desired structure and order can be found that stabilizes all plant conditions simultaneously, then an alternative algorithm for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design problem can be formulated in terms of the “Top Down” approach introduced in definition 5.4.1. Furthermore, the extension of \mathcal{H}_∞ -design algorithm to the optimal \mathcal{H}_∞ -design problem allows the definition of an algorithm that attempts to minimize the above \mathcal{H}_2 -performance cost subject to a minimally achievable \mathcal{H}_∞ -norm for each plant condition.

6.1.1 Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -Design Examples

6.1.1.1 A Second-Order Single-Plant Example

Consider the following single-plant example where the closed-loop system norms $\|T_2^1(C_0)\|_2$ and $\|T_\infty^1(C_0)\|_\infty$ can be computed explicitly. According to the system representation in (3.1) the system has the following state-space realization.

$$\dot{x}^1(t) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} x^1(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w_2^1(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w_\infty^1(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u^1(t)$$

$$z_2^1(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^1(t)$$

$$z_\infty^1(t) = \begin{pmatrix} -1 & 1 \end{pmatrix} x^1(t)$$

$$y^1(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} x^1(t).$$

This plant is detectable through $y^1(t)$ (not observable). Furthermore, it can be verified that the problem is singular and certain subsystems have invariant zeros on the $j\omega$ -axis (refer to [24] and [113]). The open-loop system is unstable and the specified controller type for this example is a static output-feedback controller $u^1(t) = C_0 y^1(t) = D_c y^1(t)$. It is easily verified that the controller will stabilize the plant for any $D_c < -1$. After some algebra we arrive at the following expressions for

the relevant norms as a function of the output–feedback gain D_c .

$$\|T_\infty^1(D_c)\|_\infty = \frac{1}{|D_c|} \quad (6.19)$$

$$\|T_2^1(D_c)\|_2 = \sqrt{\frac{1 + D_c^3}{2D_c(D_c - 1)}}. \quad (6.20)$$

The behavior of these expressions as a function of D_c is shown in Figures 6.1 and 6.2 respectively. The corresponding mixed $\mathcal{H}_2/\mathcal{H}_\infty$ –performance/robustness tradeoff characteristic for this design example is shown in Figure 6.3. Every ‘o’ in Figure 6.3 corresponds to a numerical point design while the curve connecting these points is the theoretically achievable $\mathcal{H}_2/\mathcal{H}_\infty$ –performance/robustness characteristic.

The \mathcal{H}_2 –optimal controller gain D_c^* is $D_c^* = 2.2961$ resulting in the closed–loop \mathcal{H}_2 –norm $\|T_2(D_c^*)\|_2 = 1.4486$ and a corresponding \mathcal{H}_∞ –norm $\|T_\infty(D_c^*)\|_\infty = 0.4355$. Thus for any specified \mathcal{H}_∞ –bound γ^1 satisfying $\gamma^1 > 0.4355$ the \mathcal{H}_2 –optimal controller D_c^* satisfies the specified bound $\|T_\infty^1(C_0)\|_\infty < \gamma^1$ additionally. Hence for $\gamma^1 > 0.4355$ the controller $D_c^* = 2.2961$ represents a solution to the simultaneous mixed $\mathcal{H}_2/\mathcal{H}_\infty$ –design problem discussed in chapter 2. For $\gamma^1 < 0.4355$ a trade-off between the \mathcal{H}_2 –performance and robust \mathcal{H}_∞ –stability has to be accepted. Depending on the design specifications one would then pick a point from the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ –performance/robustness tradeoff characteristic in Figure 6.3 that satisfies these specifications.

6.1.1.2 Two-Plant F15–Aircraft Model

The plant in this example represents a fourth–order model for the longitudinal dynamics of an F15–aircraft. The first plant condition represents a subsonic flight condition while the second operating condition is supersonic. The system uncertainties are in the drag coefficient (C_D) and the pitching moment coefficient (C_{M_α}). The control input is the elevator control and the four states of the model correspond to the aircraft velocity, the angle of attack, the pitch rate and the pitch attitude respectively. The reader is referred to [96] for more information on the physical parameters and the plant model itself.

According to the system representation (3.1), the system matrices describing the subsonic and supersonic operating conditions are as follows.

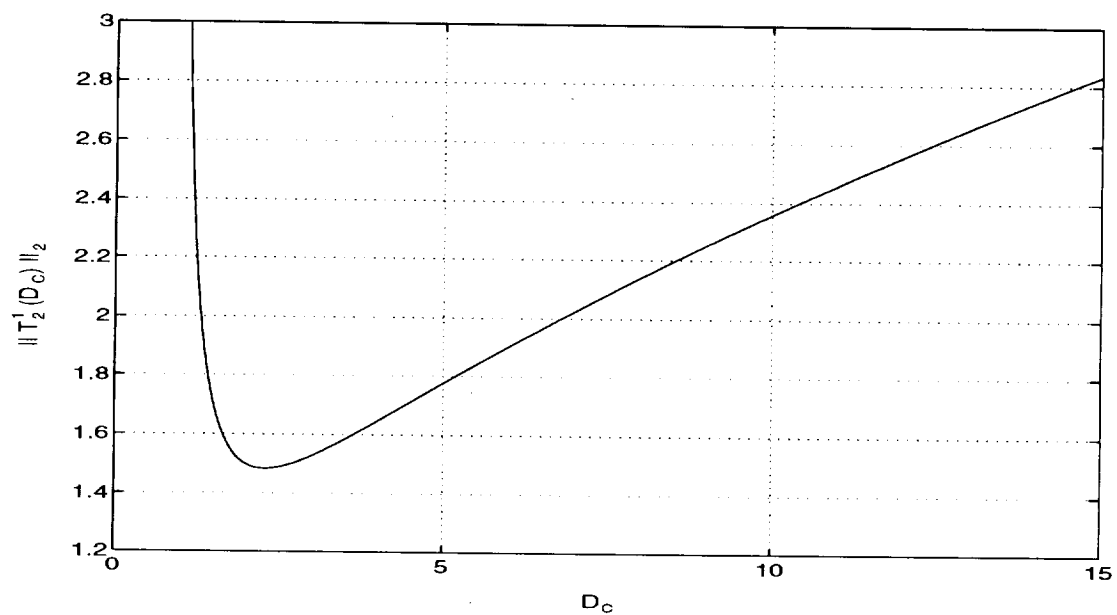


Figure 6.1: Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design example 1: $\|T_2^1(D_c)\|_2$ as a function of D_c .

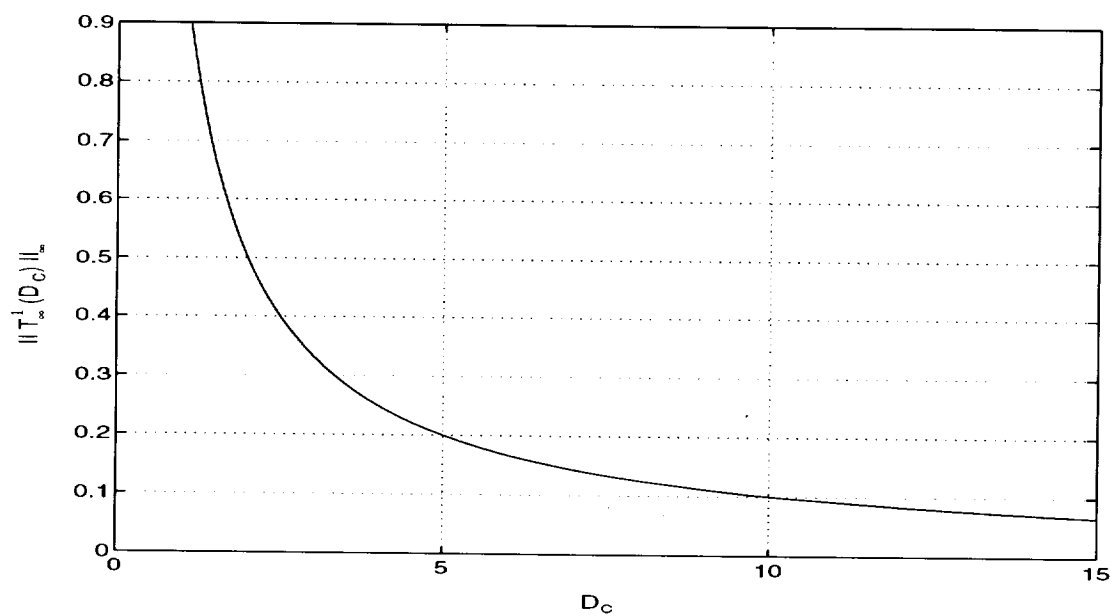


Figure 6.2: Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design example 1: $\|T_\infty^1(D_c)\|_\infty$ as a function of D_c .

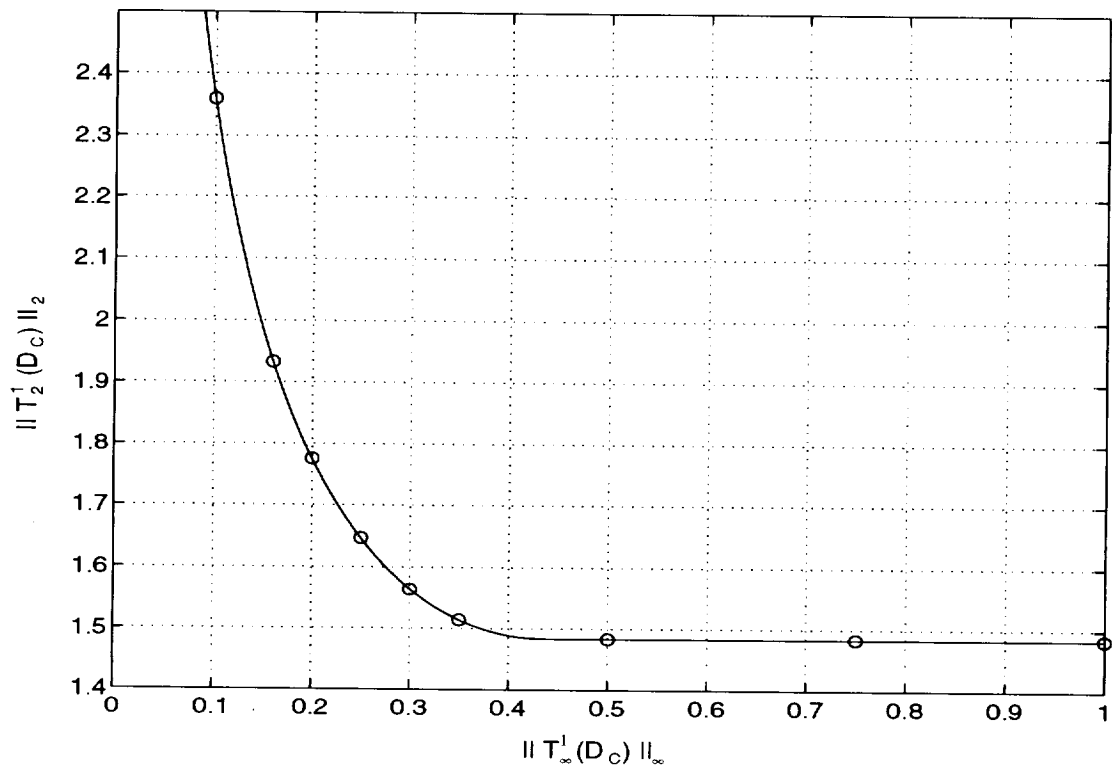


Figure 6.3: Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design example 1: $\mathcal{H}_2/\mathcal{H}_\infty$ -performance/robustness tradeoff characteristic.

First plant condition (subsonic):

$$\begin{aligned}
 A^1 &= \begin{pmatrix} 0.0082 & -25.7084 & 0 & -32.1709 \\ -0.0002 & -1.2763 & 1.0000 & 0 \\ 0.0007 & 1.0218 & -2.4052 & 0 \\ 0 & 0 & 1.0000 & 0 \end{pmatrix}, \quad B_1^1 = \begin{pmatrix} 0.0082 & 0.0462 \\ 0.0002 & 0.0023 \\ -0.0007 & -0.0018 \\ 0 & 0 \end{pmatrix}, \\
 B_2^1 &= \begin{pmatrix} -0.5585 & 0 & 0 \\ 0 & -0.2793 & 0 \\ 0 & 0.9991 & 20.9938 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_3^1 = \begin{pmatrix} -6.8094 \\ -0.1497 \\ -14.0611 \\ 0 \end{pmatrix}, \\
 C_1^1 &= \begin{pmatrix} 0.0707 & 0 & 0 & 0 \\ 0 & 0 & 0.3162 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D_{13}^1 = \begin{pmatrix} 0 \\ 0 \\ 20 \end{pmatrix}, \\
 C_2^1 &= \begin{pmatrix} 0.0147 & 0 & 0 & 0 \\ 0 & 0.0147 & 0 & 0 \\ 0 & -0.1688 & 0 & 0 \end{pmatrix}, \quad C_3^1 = I.
 \end{aligned}$$

Second plant condition (supersonic):

$$\begin{aligned}
 A^2 &= \begin{pmatrix} -0.0117 & -95.9107 & 0 & -32.1129 \\ -0.0001 & -1.8794 & 1.0000 & 0 \\ 0.0006 & -3.6163 & -3.4448 & 0 \\ 0 & 0 & 1.0000 & 0 \end{pmatrix}, \quad B_1^2 = \begin{pmatrix} 0.0117 & 0.0661 \\ 0.0001 & 0.0013 \\ -0.0006 & 0.0025 \\ 0 & 0 \end{pmatrix}, \\
 B_2^2 &= \begin{pmatrix} -0.7985 & 0 & 0 \\ 0 & -0.3993 & 0 \\ 0 & 2.0457 & 78.4635 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_3^2 = \begin{pmatrix} -25.4041 \\ -0.2204 \\ -53.4246 \\ 0 \end{pmatrix}, \\
 C_1^2 &= C_1^1, \quad D_{13}^2 = D_{13}^1, \quad C_2^2 = C_2^1, \quad C_3^2 = I.
 \end{aligned}$$

All the remaining system matrices are assumed to be zero. These matrices follow from the uncertainty description and the specified \mathcal{H}_2 -criterion defined on a weighted combination of velocity, pitch attitude and elevator control. The controller is a first-order, structurally unconstrained proper controller. By applying the presented \mathcal{H}_∞ -design method and the scheme in [64] to compute \mathcal{H}_2 -optimal controllers to both plant conditions individually (note that this involves two single-plant \mathcal{H}_2 -problems and two single-plant \mathcal{H}_∞ -problems), some preliminary information has been derived for this type of controller. These results and some open-loop information are given in table 6.1 below.

Table 6.1: F15 multi-plant mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design example: Preliminary analysis of the plants.

	Plant 1	Plant 2
Open-Loop:	Stable	Stable
Open-Loop $\ T_2^i(C_0 = 0)\ _2$	0.1068	0.0312
Open-Loop $\ T_\infty^i(C_0 = 0)\ _\infty$	23348.3	8013.3
Min. achievable $\ T_2^i(C_0)\ _2$	0.032	0.0022
Min. achievable $\ T_\infty^i(C_0)\ _\infty$	0.0563	0.0964

The design results for the two-plant mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design are shown in Figure 6.4 where the first plant condition is identified by the design points ‘*’ (lower curve) and the second plant by ‘o’ (upper curve). The weighting factors α^i were chosen to be $\alpha^1 = 1$ and $\alpha^2 = 1$. Hence both \mathcal{H}_2 -norms are weighted equally. This choice is justified as both plant conditions have roughly the same value for the minimally achievable \mathcal{H}_2 -norm. The same \mathcal{H}_∞ -bounds γ^1 and γ^2 were applied to both plant conditions for each point design, i.e., $\gamma^1 = \gamma^2$. Hence this is only a two-dimensional example out of a generally four-dimensional surface. In a mixed design for multiple plants, γ^i will provide an actual constraint for only some of the n_p operating conditions leaving the other plants unconstrained in terms of the robustness constraints. In this example the resulting $\|T_\infty^1(C_0)\|_\infty$ was always below the specified γ^1 while $\|T_\infty^2(s)\|_\infty$ stays on the specified robustness boundary for all design points. This suggests that the supersonic flight condition is the more critical operating mode for the robust

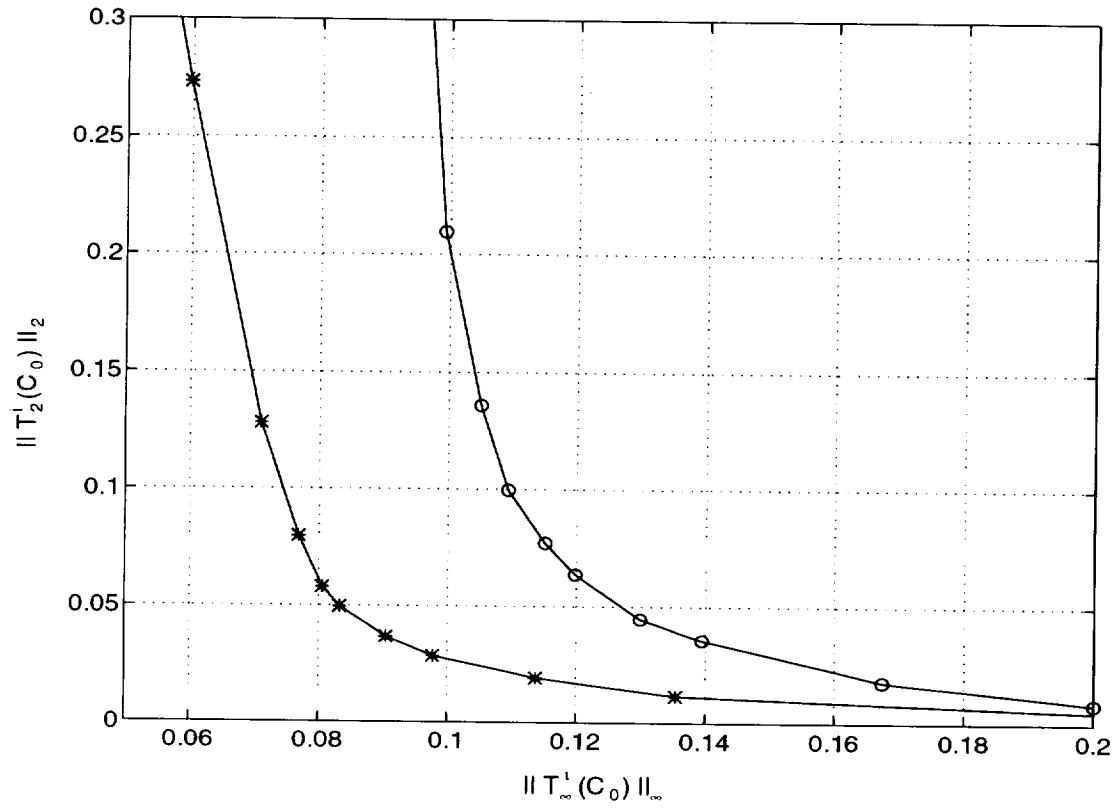


Figure 6.4: F15 multi-plant mixed $\mathcal{H}_2/\mathcal{H}_{\infty}$ -design example: $\mathcal{H}_2/\mathcal{H}_{\infty}$ -performance/robustness tradeoff characteristics for the multi-plant case.

stability problem. In general both curves exhibit the typical design tradeoffs involved in the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design.

Judging from Figure 6.4 a controller providing the the best compromise between robustness and performance (taking into account both plant conditions) is achieved with the following first-order controller:

$$A_c = \begin{pmatrix} -8.55194499 \end{pmatrix},$$

$$B_c = \begin{pmatrix} 53.16826371 & -5.65806706 & -19.24660504 & -5.53475605 \end{pmatrix},$$

$$C_c = \begin{pmatrix} 0.07340103 \end{pmatrix},$$

$$D_c = \begin{pmatrix} -0.83757176 & 15.44902573 & 1.17896427 & 8.66218328 \end{pmatrix}.$$

The closed-loop properties for both plant conditions with this particular controller are summarized in tables 6.2 and 6.3. The singular value plots for both design conditions are shown in Figures 6.5 and 6.6 respectively. Note that in this example the transfer functions $T_\infty^i(C_0, s)$ generally have three singular values (as a function of $j\omega$). In Figures 6.5 and 6.6 only the two most significant singular values are shown, the third singular value function is insignificant in comparison to the other two.

Table 6.2: F15 multi-plant mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design example: Closed-loop properties for the first plant condition.

Achieved $\ T_2^1(C_0)\ _2$:	0.0579
Achieved $\ T_\infty^1(C_0)\ _\infty$:	0.0805 (−21.8758 dB)
Closed-loop system poles:	$\lambda_1 = -0.7958$ $\lambda_2 = -4.2007$ $\lambda_3 = -7.9336$ $\lambda_{4,5} = -6.2490 \pm 14.302332j$

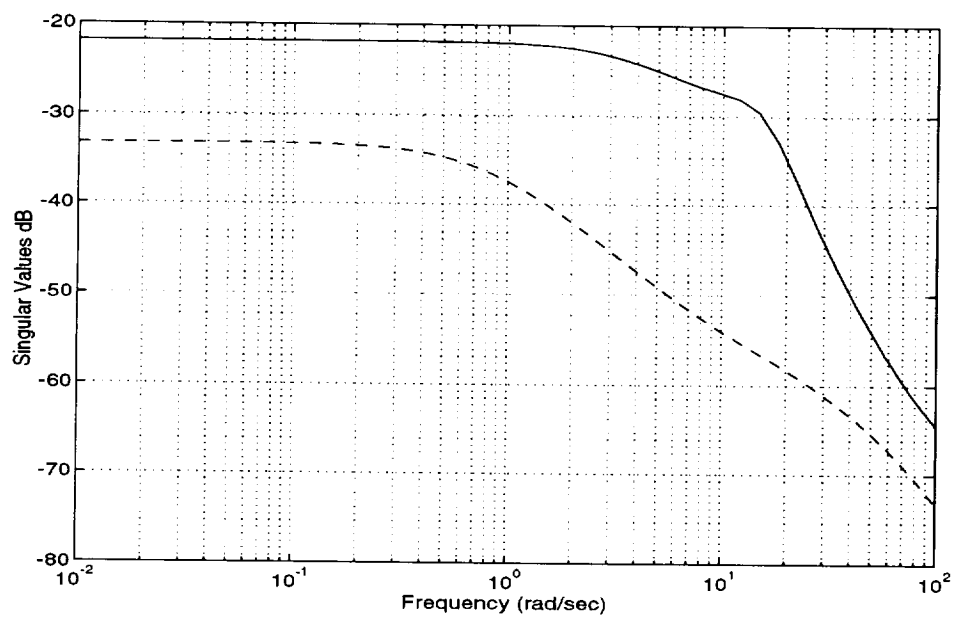


Figure 6.5: F15 multi-plant mixed $\mathcal{H}_2/\mathcal{H}_{\infty}$ -design example: Singular value plot of $T_{\infty}^1(C_0, s)$.

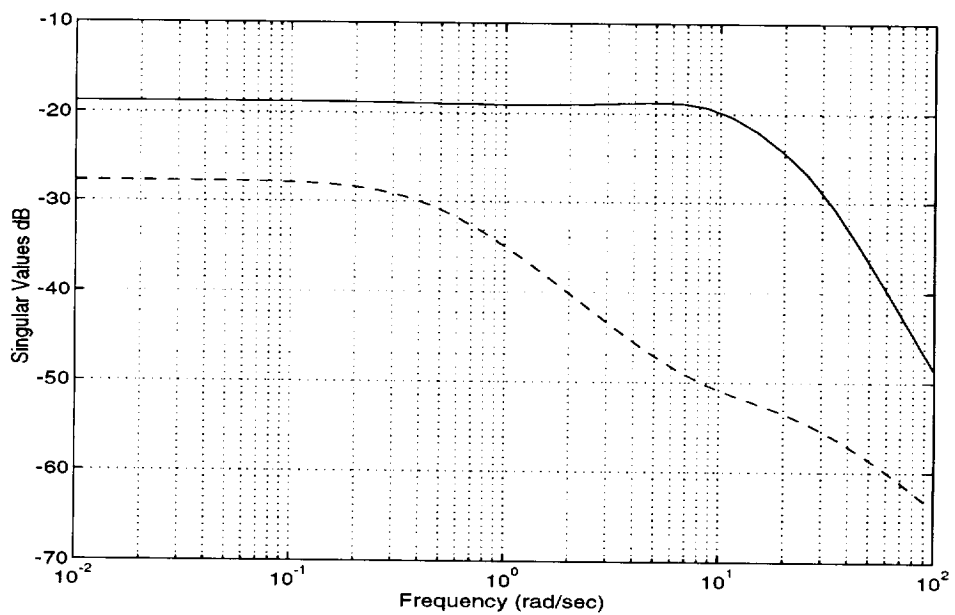


Figure 6.6: F15 multi-plant mixed $\mathcal{H}_2/\mathcal{H}_{\infty}$ -design example: Singular value plot of $T_{\infty}^2(C_0, s)$.

Table 6.3: F15 multi-plant mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design example: Closed-loop properties for the second plant condition.

Achieved $\ T_2^2(C_0)\ _2$:	0.0771
Achieved $\ T_\infty^2(C_0)\ _\infty$:	0.115 (−18.7860 dB)
Closed-loop system poles:	$\lambda_1 = -0.4658$ $\lambda_{2,3} = -7.7470 \pm 5.9286j$ $\lambda_{4,5} = -21.5206 \pm 18.3541j$

6.2 Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -Control: The Single-Plant Full State-Feedback Case

The most general \mathcal{H}_∞ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design problems are in general not convex, not even in the single plant case. However, for a very special class of single-plant problems an upper bound for the \mathcal{H}_2 -cost in conjunction with a particular controller parametrization can be formulated as a convex optimization problem. When convexity of the cost function holds (see chapter 4), this problem can then be solved via an unconstrained scalar, differentiable and convex minimization problem. Since only the single-plant case is considered here, the superscript $(*)^i$ is omitted in the following discussion. For this problem the system under consideration is assumed to have a state-space realization of the following form.

$$\Sigma_{2/\infty,op,SF} : \begin{cases} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_3 u(t) \\ z_2(t) &= C_1 x(t) + D_{13} u(t) \\ z_\infty(t) &= C_2 x(t) + D_{23} u(t) \\ y(t) &= x(t). \end{cases} \quad (6.21)$$

The system $\Sigma_{2/\infty,op,SF}$ is to satisfy the following assumptions (see [58]):

1. D_{23} has full column rank.
2. The matrix pair

$$[(I - D_{23}(D_{23}^T D_{23})^{-1} D_{23}^T) C_2, -A + B_3(D_{23}^T D_{23})^{-1} D_{23}^T C_2]$$

is observable.

The need for these assumptions will be discussed later. The dimensions of all the signals are: $x(t) \in R^{n_x}$, $w(t) \in R^{n_w}$, $z_2(t) \in R^{n_{z2}}$, $z_\infty(t) \in R^{n_{z\infty}}$, $u(t) \in R^{n_u}$. The controller is a static full-state feedback controller given by

$$u(t) = C_0 y(t) = C_0 x(t). \quad (6.22)$$

First and most importantly, there is no distinction between two disturbance signal vectors $w_2(t)$ and $w_\infty(t)$ as was done in the general case. In this formulation $w(t)$ plays a dual role as a \mathcal{H}_2 -disturbance and a \mathcal{H}_∞ -disturbance. In practical terms this implies that the disturbances $w_2(t)$ and $w_\infty(t)$ are assumed to affect the system through the same input distribution matrix B_1 . The uncertainties in this problem are still modeled by a stable, norm bounded $\Delta(s)$ -block with a feedback connection $w(s) = \Delta(s)z_\infty(s)$ and the \mathcal{H}_∞ -bound $\|\Delta(s)\|_\infty \leq \frac{1}{\gamma}$. Note that in general a direct feedthrough term from $w(t)$ to $z_\infty(t)$ can be incorporated as well. However, with preliminary transformations described in [113], this case can always be reduced to a state-space description of the form in (6.21).

With a static state-feedback controller C_0 , the closed-loop system $\Sigma_{2/\infty,cl,SF}$ is given by

$$\Sigma_{2/\infty,cl,SF} : \begin{cases} \dot{x}_{cl}(t) &= (A + B_3 C_0)x_{cl}(t) + B_{cl}w(t) &= A_{cl}x_{cl}(t) + B_1 w(t) \\ z_2(t) &= (C_1 + D_{13}C_0)x_{cl}(t) &= C_{cl,2}x_{cl}(t) \\ z_\infty(t) &= (C_2 + D_{23}C_0)x_{cl}(t) &= C_{cl,\infty}x_{cl}(t). \end{cases} \quad (6.23)$$

For this type of systems, the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design problem is essentially the same as the previously considered problem, only that the \mathcal{H}_2 -cost is now defined for the closed-loop transfer function $T_2(s)$ from $w(s)$ to $z_2(s)$ and the corresponding \mathcal{H}_∞ -constraint on the closed-loop transfer function $T_\infty(s)$ from $w(s)$ to $z_\infty(s)$. With these definitions the problem statement is similar to that stated in definition 3.2.1 for the single-plant case with $n_p = 1$, and is omitted here. Assuming that a state-feedback gain C_0 stabilizes the closed-loop system then the \mathcal{H}_2 -cost is

$$\|T_2(C_0)\|_2^2 = \text{Trace}\{ C_{cl,2} L_c C_{cl,2}^T \} \quad (6.24)$$

where L_c solves

$$L_c A_{cl}^T + A_{cl} L_c + B_{cl} B_{cl}^T = 0. \quad (6.25)$$

The \mathcal{H}_∞ -constraint on $T_\infty(s)$ such that $\|T_\infty(C_0)\|_\infty < \gamma$ for a specified γ can be represented by the ARI

$$ARI_{C,SF}(C_0, X, \gamma) < 0 \quad (6.26)$$

$$ARI_{C,SF}(C_0, X, \gamma) := XA_{cl}^T + A_{cl}X + \gamma^{-2}XC_{cl,2}^TC_{cl,2}X + B_{cl}B_{cl}^T \quad (6.27)$$

subject to $X > 0$. By subtracting (6.25) from (6.27) it immediately follows that $X \geq L_c$ if $ARI_{C,SF}(C_0, X, \gamma) < 0$ is satisfied and hence the cost $Trace\{C_{cl,2}XC_{cl,2}^T\}$ is an upper bound for the \mathcal{H}_2 -cost $\|T_2(C_0)\|_2^2$ if (6.26) is satisfied. This fact was first reported by Bernstein et. al. and has been investigated thoroughly in [6], [7], [47]. However, there is no reason to consider an upper bound to the \mathcal{H}_2 -cost if no additional advantage can be derived.

Now consider a controller factorization of the form $C_0 = WX^{-1}$ where W is a real matrix $W \in R^{n_u \times n_x}$. For this factorization the following abbreviation is introduced.

$$ARI_{C,SF}(C_0 = WX^{-1}, X, \gamma) = ARI_{C,SF}(W, X, \gamma).$$

With such a factorization the following results have been derived in [55] and [56].

Lemma 6.2.1 ([55], [6])

Consider the closed-loop system $\Sigma_{2/\infty,cl,SF}$ with a fixed \mathcal{H}_∞ -bound γ and let $C_0 = WX^{-1}$ with $X = X^T > 0$ and $W \in R^{n_u \times n_x}$, then the following holds:

1. For constant γ the matrix function $ARI_{C,SF}(W, X, \gamma) : (W, X) \rightarrow R^{n_x \times n_x}$

$$ARI_{C,SF}(W, X, \gamma) = XA_{cl}^T + A_{cl}X + \gamma^{-2}XC_{cl,2}^TC_{cl,2}X + B_{cl}B_{cl}^T \quad (6.28)$$

$$= X[A + B_3WX^{-1}]^T + [A + B_3WX^{-1}]X \quad (6.29)$$

$$\begin{aligned} & + \gamma^{-2}X[C_2 + D_{23}WX^{-1}]^T[C_2 + D_{23}WX^{-1}]X + B_1B_1^T \\ & = XA^T + AX + B_3W + W^TB_3^T + B_1B_1^T \\ & \quad + \gamma^{-2}[C_2X + D_{23}W]^T[C_2X + D_{23}W] \end{aligned} \quad (6.30)$$

is jointly convex in X and W . Furthermore, there exists a static state-feedback $C_0 = WX^{-1}$ such that $\|T_\infty(C_0)\|_\infty < \gamma$ if and only if there are $X = X^T > 0$ and W such that $ARI_{C,SF}(W, X, \gamma) < 0$.

2. The scalar quantity

$$J_{2,SF}(W, X) = \text{Trace}\{[C_1 + D_{13}WX^{-1}]X[C_1 + D_{13}WX^{-1}]^T\} \quad (6.31)$$

is jointly convex in X and W . Moreover, if $X = X^T > 0$ and W exist such that $ARI_{C,SF}(W, X, \gamma) < 0$, then

$$J_{2,SF}(W, X) \geq \|T_2(C_0 = WX^{-1})\|_2^2 = \|T_2(W, X)\|_2^2. \quad (6.32)$$

Two important remarks can be made at this point. First, no means are available to determine the gap between the upper bound $J_{2,SF}(W, X)$ and the actual \mathcal{H}_2 -cost $\|T_2(C_0)\|_2^2$ and hence the upper bound can be very conservative. Secondly, although the above lemma gives an if-and-only-if condition between the existence of a controller $C_0 = WX^{-1}$ that satisfies the specified \mathcal{H}_∞ -constraint, *not all* possible controllers that satisfy the \mathcal{H}_∞ -constraint are included in the class of possible solutions X and W that satisfy the \mathcal{H}_∞ -constraint.

To illustrate this fact, assume that the controller $C_0^* = W^*(X^{-1})^*$ satisfies the \mathcal{H}_∞ -constraint. Then $C_0^* = [\beta W^*][\frac{1}{\beta}(X^{-1})^*]$ satisfies the \mathcal{H}_∞ -constraint as well for any scalar $\beta \neq 0$. However, there will always be a positive β such that condition 4 in theorem 2.2.1 is violated and hence $\frac{1}{\beta}(X^{-1})^*$ and $[\beta W^*]$ do not satisfy $ARI_{C,SF}(W, X, \gamma)$ any more. Hence, despite the necessary and sufficient condition for the existence of a \mathcal{H}_∞ -suboptimal controller, not all possible controller characterizations in terms of X and W are included in lemma 6.2.1. Thus, if additional performance measures are to be achieved based on these individual controller components, the above \mathcal{H}_∞ -bound characterization is conservative. Note that this scheme is applicable only to the single-plant problem as the controller parametrization directly depends on the solution X of $ARI_{C,SF}(W, X, \gamma) < 0$. Despite its conservatism, the formulation of an upper bound mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -problem in terms of (6.32) and (6.29) has the advantage that it results in a convex optimization problem to which the penalty cost function approach is applicable. Following the framework established so far, this problem can now be formulated as follows.

Definition 6.2.1

For the open-loop system $\Sigma_{2/\infty,op,SF}$ in (6.21) and a prespecified \mathcal{H}_∞ -bound γ , find two matrices W^* and $X^* = (X^*)^T > 0$ and a corresponding static state-feedback controller $C_0^* = W^*(X^*)^{-1}$ that solves the minimization problem

$$J_{2,SF}(W^*, X^*) = \min_{W, X} J_{2,SF}(W, X) \quad (6.33)$$

subject to

$$\lim_{\substack{t_{f1} \rightarrow \infty \\ t_{f3} \rightarrow \infty}} J_{\infty,SF}(W^*, X^*, \gamma, t_{f1}, t_{f3}) = 0, \quad (6.34)$$

where

$$J_{\infty,SF}(W, X, \gamma, t_{f1}, t_{f3}) = \text{Trace} \{ e^{ARI_{C,SF}(W,X,\gamma)t_{f1}} + e^{-Xt_{f3}} \}. \quad (6.35)$$

Some interesting results have been derived for this type of problem. Most importantly, in [58] it has been shown that for the general mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -problem a dynamic state-feedback controller C_0 will not outperform a static state-feedback controller. That means that we can restrict our attention to the class of static full-state feedback controllers. Note, however, that this is not necessarily true for the simultaneous mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -problem. \mathcal{H}_2 -optimal controllers that additionally satisfy a specified \mathcal{H}_∞ -constraint on $T_\infty(s)$ may in general be dynamic for this problem ([92]). In this work the attention is restricted to the static case and hence to the general mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -problem. Extensions to the general dynamic case are easily incorporated and will form a convex optimization problem as well.

The design problem can be solved numerically in the same way as the multi-plant mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -problem. Note in particular, that for any given t_{f1} and t_{f3} , the resulting minimization problems are convex as $J_{2,SF}(W, X)$, $ARI_{C,SF}(W, X, \gamma)$ and $-X$ are jointly convex in W and X and hence, with the results in chapter 2, $J_{\infty,SF}(W, X, \gamma, t_{f1}, t_{f3})$ is jointly convex in W and X as well. As the sum of convex functions is convex, overall convexity follows. Thus, using the penalty function algorithm, the upper bound mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -problem can be solved via a series of unconstrained convex minimization problems. Note also that in this case the closed-loop system may be unstable during intermediate phases of the algorithm as the \mathcal{H}_2 -cost is not computed in terms of a Lyapunov solution. As before, once W^* and $X^* = (X^*)^T > 0$ have been found such that $ARI_{C,SF}(W, X, \gamma) < 0$, then closed-loop

stability automatically follows. The convexity property holds for any factorization of X imposing $X = X^T$ as discussed in chapter 5. Furthermore, the assumptions imposed on the open-loop system assure that the solutions W and X are finite and simultaneously avoid local minima that do not satisfy the \mathcal{H}_∞ -bounds. In general these constraints can be removed by adding an auxiliary cost to ensure bounded solutions of W and X (see chapter 5).

6.2.1 Full State-Feedback Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -Design Example

The example plant is the 4th-order system used in [101]. It represents the scaled subsystem of the lateral dynamics of a B-767 aircraft with uncertain entries in the open-loop A -matrix. The state-space matrices for this plant are as follows.

$$A = \begin{pmatrix} -0.0168 & 0.1121 & 0.0003 & -0.5608 \\ -0.0164 & -0.7771 & 0.9945 & 0.0015 \\ -0.0417 & -3.6595 & -0.9544 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} -0.0243 \\ -0.0634 \\ -3.6942 \\ 0 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}, \quad D_{13} = 1,$$

$$C_2 = \begin{pmatrix} 0.01 & 0 & 0.01 & 0 \end{pmatrix}, \quad D_{23} = 0.01, \quad C_3 = I.$$

All other matrices are assumed to be zero. The open-loop system is stable, the relevant open-loop norms are $\|T_\infty(C_0)\|_\infty = 7.4826$ and $\|T_2(C_0)\|_2 = 0.9260$ respectively, the open-loop subsystem $T_\infty(C_0, s)$ has zeros in the right-half plane, the minimally achievable $\|T_\infty(C_0)\|_\infty$ is 0.007 and the minimally achievable \mathcal{H}_2 -norm $\|T_2(C_0)\|_2$ is 0.0078 when the \mathcal{H}_∞ and \mathcal{H}_2 -problems are solved independently for the state-feedback case.

The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -performance/robustness characteristic for this state-feedback case has the same interpretations and properties as that in the general case output-

feedback case. This tradeoff is shown in figure 6.7. A good tradeoff between the \mathcal{H}_2 -performance and stability robustness can be achieved with the following state-feedback gain matrix. The matrices W and X from which D_c is computed are

$$W = \begin{pmatrix} -0.73383998 & -0.28674417 & -0.22309747 & 0.67626086 \end{pmatrix}$$

$$X = \begin{pmatrix} 13.77087597 & 0.60885959 & 0.26514956 & 2.21194320 \\ 0.60885959 & 0.24309849 & 0.18079149 & -0.52517534 \\ 0.26514956 & 0.18079149 & 0.15950869 & -0.52847260 \\ 2.21194320 & -0.52517534 & -0.52847260 & 2.61509640 \end{pmatrix}.$$

From these matrices W and X the controller gain D_c then follows from $D_c = WX^{-1}$.

$$D_c = \begin{pmatrix} -0.04438848 & -0.83242054 & 0.13897734 & 0.15705908 \end{pmatrix}. \quad (6.36)$$

The closed-loop properties for this particular controller are summarized in table 6.4 and the corresponding singular value plot of the closed-loop system is shown in Figure 6.8.

Table 6.4: B-767 mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design example, full state-feedback: Closed-loop properties.

Achieved $\ T_2(C_0)\ _2$:	0.2335
Achieved $\ T_\infty(C_0)\ _\infty$:	0.0921 (−20.7103 dB)
Closed-loop system poles:	$\lambda_1 = -0.4653$ $\lambda_2 = -7.5807$ $\lambda_{3,4} = -0.6025 \pm 0.7529j$

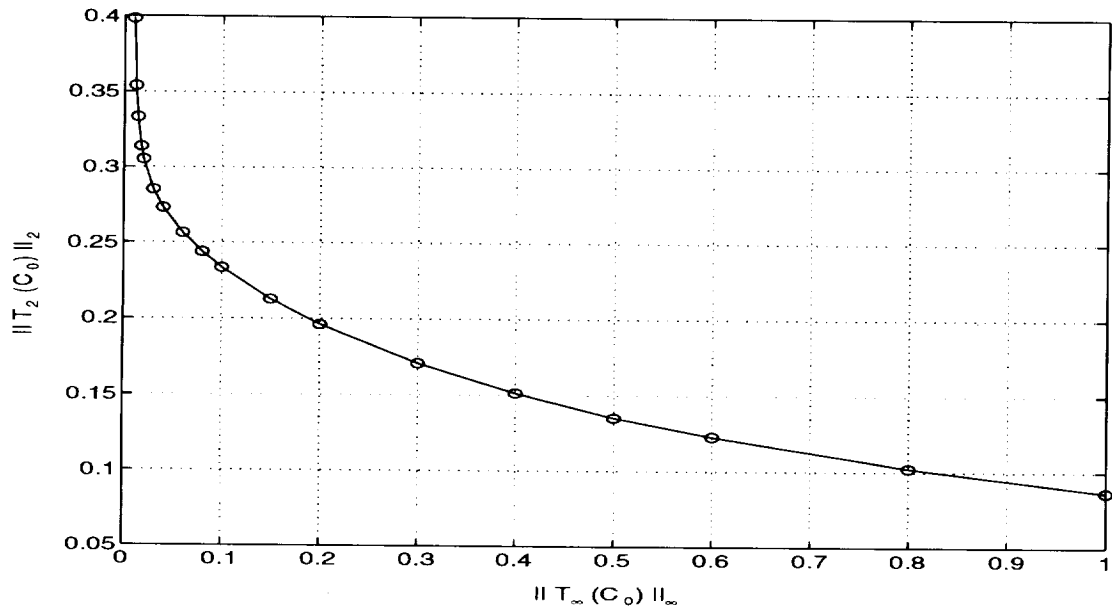


Figure 6.7: B-767 mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design example, full state-feedback: $\mathcal{H}_2/\mathcal{H}_\infty$ -performance/robustness characteristic.

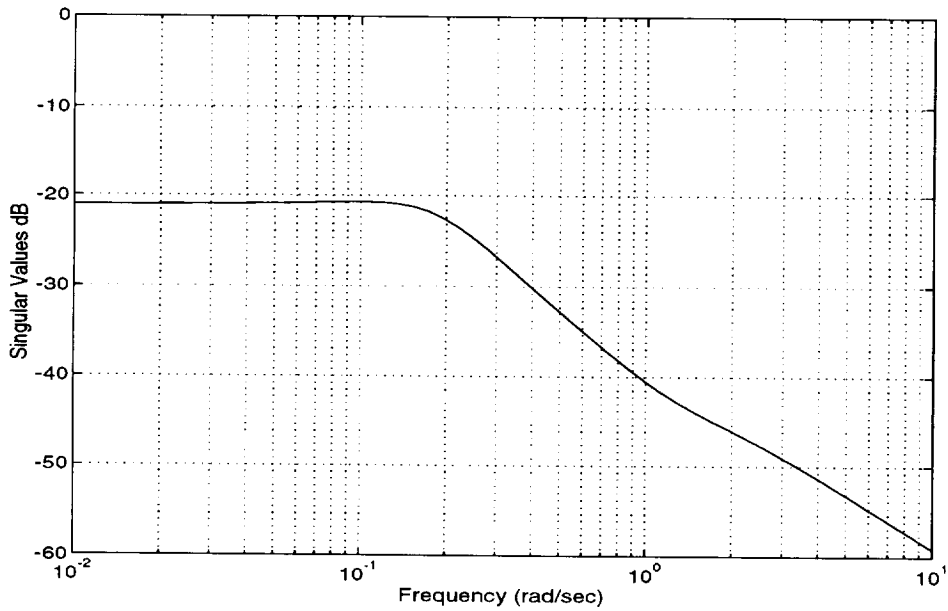


Figure 6.8: B-767 mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design example, full state-feedback: Singular value plot of $T_\infty(C_0, s)$.

Chapter 7

CONCLUDING REMARKS

In this work a new approach for the general \mathcal{H}_∞ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design problems has been presented. The approach is based on the representation of \mathcal{H}_∞ -constraints in terms of matrix inequalities. By the use of a new type of scalar cost function these matrix constraints have been converted to scalar differentiable constraints that can be appended to any performance-oriented optimization problem. This formulation makes the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design problem amenable to a gradient-based solution. The developed scheme can incorporate features such as fixed-structure/fixed-order controllers, it can accommodate multiple operating conditions and places only a minimal set of system assumptions on the open-loop systems. In particular it does not impose assumptions on the system zeros or orthogonality conditions as in previous formulations and provides a general framework for the \mathcal{H}_∞ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design problems.

In general there is a variety of possible ways to formulate the optimization problems associated with the \mathcal{H}_∞ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design problems. Some of them have been discussed so as to point out alternative routes for posing the minimization problems. In this work a single cost function has been defined that contains all the performance cost functionals as well as the penalty/barrier functions associated with the \mathcal{H}_∞ -constraints. With this overall cost function the \mathcal{H}_∞ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design problems reduce to a sequence of unconstrained minimization problems. Corresponding gradient expressions for all the cost functions have been derived in the appendices. The analysis of these partial gradients has provided valuable information on the existence of local minima that do not satisfy the desired \mathcal{H}_∞ -constraints. Possible ways to exclude such local minima have been presented and discussed.

The defined trace-type cost function has the property that it is convex if the underlying matrix inequality is convex. This fact allows the formulation of a differentiable convex optimization problem for the full state-feedback single-plant mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -problem applied to a special class of systems. For this class of systems a

convex upper bound for the \mathcal{H}_2 -cost is considered subject to a set of convex matrix inequalities representing the \mathcal{H}_∞ -constraint. However, in the general multi-plant case (or the single-plant case with the exact \mathcal{H}_2 -norm as performance measure) the \mathcal{H}_2 -cost and the matrix inequalities enforcing the \mathcal{H}_∞ -constraints are not convex. In this case one can devise alternative optimization schemes based only on the eigenvalues that violate the considered matrix inequality constraints and/or on finite-time \mathcal{H}_2 -performance costs [64]. The contents of appendix A is intended to serve as a basis for the derivation of explicit gradient expressions for such schemes as well as for other modifications to the presented formulation of the \mathcal{H}_∞ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design problems. The included examples are non-trivial and provide valuable tests for the capabilities of the presented scheme.

In the appendices C and D it is shown that the reformulation of the \mathcal{H}_∞ -constraints in terms of the defined cost function can be applied to \mathcal{H}_∞ -constrained control problems where the “performance” criterion is not an \mathcal{H}_2 -norm.

In appendix C the performance criterion is identical to the Froebenius norm of the static state-feedback gain matrix while the \mathcal{H}_∞ -constraint guarantees robust stability. Such a performance measure has implications for the noise-sensitivity as well the control effort in the closed-loop system.

In appendix D performance correlates to time-domain constraints that are reformulated in terms of convex scalar constraints on the closed-loop system matrix and the state-feedback gain matrix. Furthermore, the problem formulation for the \mathcal{H}_∞ -constrained control problem with time-domain constraints illustrates the applicability of the presented scheme to discrete-time \mathcal{H}_∞ -constraints. Also, for this problem the LMI-characterization of \mathcal{H}_∞ -constraints has been utilized. Corresponding gradients are stated and hence the use of the penalty cost function for the enforcement of block-structured matrix inequalities is exemplified.

The design methodologies in appendices C and D show that the trace-type cost function not only allows the incorporation of robust stability in terms of \mathcal{H}_∞ -constraints in any (gradient-based) performance-oriented design problem. These applications also illustrate that the cost function is applicable to reformulate other (convex) symmetric matrix inequality constraints as (convex) scalar constraints. It is hoped that this capability opens up venues to solve problems other than these considered here.

Chapter 8

EXTENSIONS AND FUTURE RESEARCH

The defined cost function allows the reformulation of the \mathcal{H}_∞ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design problems as differentiable constrained or unconstrained optimization problems. To avoid local minima, however, it is necessary to modify the design equations or to introduce additional constraints. The use of other ARI-forms (see (2.22) – (2.24), (2.32) or (2.26)), the reformulation of \mathcal{H}_∞ -constraints in terms of LMI's (see lemma 2.2.5) or a combination of some of these inequality constraints may form a computational framework that eliminates the possibility of undesirable local minima without additional constraints. That is, a matrix inequality representation of a \mathcal{H}_∞ -constraint (or a combination of such matrix inequality constraints) is sought such that all the partial gradients of the functional representation in terms of the defined cost function are zero if and only if the corresponding \mathcal{H}_∞ -constraint is satisfied. A thorough analysis of the corresponding gradient expressions should prove valuable for this task.

Although *explicit* second-order gradients could not be found for the defined trace-type cost functions, using (A.80) it appears to be possible to find *computable* expressions for the second-order derivatives (see [64]). This should improve the convergence of the nonlinear parameter optimization considerably. Furthermore, for an efficient solution of the arising optimization problems associated with the \mathcal{H}_∞ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design problems “dedicated” software is necessary that takes into account the specific characteristics of the defined minimization problems. In particular specialized C or FORTRAN code combined with contemporary nonlinear optimization software should provide acceptable speed for the function evaluations and gradient computations as well as the search direction updates.

So far all \mathcal{H}_∞ -design methods depend – in one way or another – on the solution of ARE's or ARI's (LMI's) which requires the introduction of additional design parameters other than the controller entries, namely the sought-after solution to these ARE's, ARI's or LMI's. This necessity increases the number of optimization variables if a gradient-based parameter optimization scheme is used to solve the design

problem. The eigenvalue structure of the Hamiltonian M_γ in 2.2.2 on the other hand provides a means to test whether a closed-loop \mathcal{H}_∞ -bound is satisfied or not, based only on the closed-loop system matrices and hence dependent only on the controller C_0 . Hence the exploitation of the special eigenvalue distribution of M_γ may provide a unique tool to solve suboptimal \mathcal{H}_∞ -problems with a considerably lower number of design variables. This fact justifies further research along these lines.

Possible extensions of the presented research include the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -estimation problem as well as μ -design and ultimately mixed \mathcal{H}_2/μ -design philosophies if the uncertainty structure is known and can be exploited.

The μ -design problem can be viewed as a scaled \mathcal{H}_∞ -problem with additional scales $D(s)$ which can be considered as additional optimization variables. Assuming a state-space representation for the scales $D(s)$, one can form a state-space representation of the closed-loop system as a function of the controller C_0 and the system matrices describing $D(s)$. The μ -design problem is then equivalent to an \mathcal{H}_∞ -bound on this (scaled) closed-loop system and thus to a matrix inequality. Hence the extension of the presented \mathcal{H}_∞ -design philosophy to the μ -design problem follows naturally from the considerations in this work. Once this problem is solved, the mixed \mathcal{H}_2/μ -design objective is approached in the same way as the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -design presented here. Note that this formulation allows the solution of the μ and mixed \mathcal{H}_2/μ -design problems in terms of a sequence of minimization problems that are guaranteed to converge (though not necessarily to the global optimum). Furthermore, requirements for fixed-order controllers can be accommodated. Present design algorithms cannot guarantee convergence and often result in controllers with extremely large order, a property that is not tolerable for practical control applications.

Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -estimation on the other hand is the natural counterpart to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -control problem. Both LQG and Kalman filtering problems require the plant to be known exactly, uncertainties as considered in this work cannot be incorporated. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -estimators could prove to be important tools for design problems where internal states of the system need to be estimated in an “ $\mathcal{H}_2/\mathcal{H}_\infty$ -optimal” sense.

Finally, due to the versatility of the defined cost function it is expected that future research will show that other control problems can be posed as matrix inequalities which in turn can be solved by the presented penalty/barrier function approach.

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Appendix A

AUXILIARY MATRIX RESULTS

A.1 General Matrix Results

This section provides matrix results that are of importance to the proofs in this report. Most of the lemmas are stated without proof. The proofs can be found in the respective references.

Lemma A.1.1 (Schur Complement)

Let G, H and L be real symmetric matrices, then

$$\begin{pmatrix} G & L \\ L^T & H \end{pmatrix} = \begin{pmatrix} G & L \\ L^T & H \end{pmatrix}^T > 0 \quad (\text{A.1})$$

if and only if

$$H > 0, \quad (\text{A.2})$$

$$G - LH^{-1}L^T > 0. \quad (\text{A.3})$$

The Schur complement formula is the basis for the transformation of ARI-type \mathcal{H}_∞ -characterizations into any block structured inequality constraint such as in lemma 2.2.4 and lemma 2.2.5.

Theorem A.1.1 (Weyl's Theorem, [49], p.181)

Let $G, H \in R^{n \times n}$ be Hermitian matrices, let the eigenvalues of G, H and $G + H$ be arranged in the following order

$$\lambda_1(G) \leq \lambda_2(G) \leq \dots \leq \lambda_n(G) = \bar{\lambda}(G),$$

$$\lambda_1(H) \leq \lambda_2(H) \leq \dots \leq \lambda_n(H) = \bar{\lambda}(H), \text{ and}$$

$$\lambda_1(G + H) \leq \lambda_2(G + H) \leq \dots \leq \lambda_n(G + H) = \bar{\lambda}(G + H),$$

then

$$\lambda_i(G + H) \leq \lambda_i(G) + \bar{\lambda}(H) \quad (\text{A.4})$$

for $i = 1, 2, \dots, n$.

In particular we have

$$\bar{\lambda}(G + H) \leq \bar{\lambda}(G) + \bar{\lambda}(H). \quad (\text{A.5})$$

Furthermore, for $H \leq 0$ we have the monotonicity properties

$$\lambda_i(G + H) \leq \lambda_i(G) \quad (\text{A.6})$$

$$\bar{\lambda}(G + H) \leq \bar{\lambda}(G) \quad (\text{A.7})$$

and, for $H \geq 0$

$$\lambda_i(G) \leq \lambda_i(G + H). \quad (\text{A.8})$$

for $i = 1, 2, \dots, n$.

Lemma A.1.2 ([133], p.630)

Let $G, H \in R^{n \times n}$ be two real symmetric matrices such that $G \geq 0$ and $H \geq 0$, then

$$\text{Trace}(GH) \leq \bar{\lambda}(G)\text{Trace}(H). \quad (\text{A.9})$$

Lemma A.1.3

Consider two real symmetric matrices G and H , then

$$G \leq H \implies \text{Trace}[e^G] \leq \text{Trace}[e^H]. \quad (\text{A.10})$$

Lemma A.1.4 ([50])

Consider two real symmetric positive-semidefinite G and H and $\alpha \in (0, 1)$, then

$$\text{Trace}\{G^\alpha H^{(1-\alpha)}\} \leq [\text{Trace}(G)]^\alpha [\text{Trace}(H)]^{(1-\alpha)}. \quad (\text{A.11})$$

This lemma is a direct consequence of Weyl's theorem and the continuity property of eigenvalues of hermitian matrices.

Lemma A.1.5 (Arithmetic–Geometric Mean Inequality, [3])

Let a and b be two non-negative scalars, then

$$a^\alpha b^{(1-\alpha)} \leq \alpha a + (1 - \alpha)b \quad (\text{A.12})$$

for every $\alpha \in (0, 1)$.

Lemmas A.1.4 and A.1.5 can be combined to yield the following result.

Lemma A.1.6 ([50])

Consider two real symmetric matrices G and H , $\alpha \in (0, 1)$ and a real positive scalar t_f , then

$$\text{Trace}\{e^{[\alpha G + (1-\alpha)H]t_f}\} \leq \alpha \text{Trace}(e^{Gt_f}) + (1 - \alpha) \text{Trace}(e^{Ht_f}). \quad (\text{A.13})$$

Lemma A.1.7 ([50], [67])

Consider the matrix-valued function $M(G) = G^T H G$ with $H = H^T \geq 0$ and $\alpha \in [0, 1]$ and G, H of compatible dimensions such that $M(G)$ is defined. Then

$$\begin{aligned} \alpha M(G_1) + (1 - \alpha)M(G_2) - M[\alpha G_1 + (1 - \alpha)G_2] = \\ \alpha(1 - \alpha)(G_1 - G_2)^T (G_1 - G_2) \geq 0 \end{aligned} \quad (\text{A.14})$$

for two matrices G_1 and G_2 of compatible dimensions.

Lemma A.1.8 ([50], [67])

Consider the matrix-valued function $M(G) = G^{-1}$ for $G = G^T > 0$. Let G_1 and G_2 be two matrices satisfying $G_1 = G_1^T > 0$ and $G_2 = G_2^T > 0$, then

$$\begin{aligned} \alpha M(G_1) + (1 - \alpha)M(G_2) - M[\alpha G_1 + (1 - \alpha)G_2] = \\ \alpha(1 - \alpha)G_1^{-1}(G_2 - G_1)G_2^{-1}M[\alpha G_1 + (1 - \alpha)G_2]^{-1}G_1^{-1}(G_2 - G_1)G_2^{-1} \geq 0. \end{aligned} \quad (\text{A.15})$$

Note that lemmas A.1.7 and A.1.8 imply that $\text{Trace}\{G^T H G\}$ is convex in G for $H = H^T \geq 0$ and that $\text{Trace}\{G^{-1}\}$ is convex on the set of symmetric positive-definite matrices. Next some convexity results are presented that are utilized in appendices C and D.

Theorem A.1.2

For real matrices $G = G^T > 0$ and real positive scalars τ the scalar-valued function $f(\tau, G)$ given by

$$f(\tau, G) = \tau^2 \bar{\lambda}(G^{-1}) \quad (\text{A.16})$$

is jointly convex on G and τ .

Proof: The proof utilizes results in [58] and is very similar to that. As $f(\alpha\tau, \alpha G) = \alpha f(\tau, G)$ we only have to show that

$$f(\tau_1 + \tau_2, G_1 + G_2) \leq f(\tau_1, G_1) + f(\tau_2, G_2) \quad (\text{A.17})$$

for two arbitrary real symmetric positive-definite matrices G_1 and G_2 and two positive scalars τ_1 and τ_2 . Let T be a nonsingular matrix such that

$$T^T G_1 T = \Lambda_1 = \text{diag}(\lambda_{1,i}), \quad i = 1, 2, \dots, n \quad (\text{A.18})$$

$$T^T G_2 T = \Lambda_2 = \text{diag}(\lambda_{2,i}), \quad i = 1, 2, \dots, n. \quad (\text{A.19})$$

Such a matrix T exists for positive-definite matrices G_1 and G_2 (see [58]).

$$f(\tau_1 + \tau_2, G_1 + G_2) = \bar{\lambda}[(\tau_1 + \tau_2)^2 (G_1 + G_2)^{-1}] \quad (\text{A.20})$$

$$= \bar{\lambda}[T^{-1} \{ \text{diag}(\frac{(\tau_1 + \tau_2)^2}{\lambda_{1,i} + \lambda_{2,i}}) \} T^{-T}] \quad (\text{A.21})$$

$$= \bar{\lambda}[T^{-1} \{ \text{diag}(\frac{\tau_1^2}{\lambda_{1,i}} + \frac{\tau_2^2}{\lambda_{2,i}} + \psi_i) \} T^{-T}] \quad (\text{A.22})$$

$$= \bar{\lambda}[\tau_1^2 T^{-1} \Lambda_1^{-1} T^{-T} + \tau_2^2 T^{-1} \Lambda_2^{-1} T^{-T} \quad (\text{A.23})$$

$$+ T^{-1} \text{diag}(\psi_i) T^{-T}] \quad (\text{A.24})$$

where $\psi_i = -\frac{(\tau_1 \lambda_{2,i} - \tau_2 \lambda_{1,i})^2}{(\lambda_{1,i} + \lambda_{2,i}) \lambda_{1,i} \lambda_{2,i}}$, $i = 1, 2, \dots, n$.

Thus

$$f(\tau_1 + \tau_2, G_1 + G_2) = \bar{\lambda}[\tau_1^2 G_1^{-1} + \tau_2^2 G_2^{-1} + Q] \quad (\text{A.25})$$

for $Q = T^{-1} \text{diag}(\psi_i) T^{-T} \leq 0$ and hence

$$f(\tau_1 + \tau_2, G_1 + G_2) \leq \bar{\lambda}(\tau_1^2 G_1^{-1} + \tau_2^2 G_2^{-1}) \quad (\text{A.26})$$

$$\leq \bar{\lambda}(\tau_1^2 G_1^{-1}) + \bar{\lambda}(\tau_2^2 G_2^{-1}) \quad (\text{A.27})$$

$$= f(\tau_1, G_1) + f(\tau_2, G_2) \quad (\text{A.28})$$

which implies joint convexity in τ and G for $G = G^T > 0$. The inequalities follow immediately from the above Lemmas and Weyl's Theorem. An alternative proof can be constructed using Fischer's min-max theorem (see [67]). ■

Theorem A.1.3

For real matrices $G = G^T > 0$ and real positive scalars τ the scalar-valued function $f(\tau, G)$ given by

$$f(\tau, G) = \text{Trace}(\tau^2 G^{-1}) \quad (\text{A.29})$$

is jointly convex on G and τ .

Proof: A proof can be constructed using the same tools as above. It is essentially equivalent to the proof of Lemma 4.4 in [58] and is omitted here. ■

Theorem A.1.4

For real matrices $G = G^T > 0$ and real positive scalars τ the matrix-valued function $M(\tau, G)$ given by

$$M(\tau, G) = \frac{1}{\tau^2} I - G. \quad (\text{A.30})$$

Then the matrix function $M(\tau, G)$ is jointly convex on G and τ .

Proof: $M(\tau, G)$ is affine in G . $\frac{1}{\tau^2}$ is a strictly monotonically decreasing function for all $\tau > 0$. Joint convexity on G and τ follows immediately. ■

A.2 Gradient-Related Matrix Results

In this section various gradient-based matrix results are stated that are required in the derivation of explicit gradients for the cost functions considered in this work. The notation concerning gradients with respect to matrices, vectors or scalars is standard and corresponds to that in [67], [133] and references therein. These references as well as [90] are excellent sources for further results related to gradient computations. In the following scalar functions $f(K)$ and matrix-valued functions $M(K)$ of a real parameter matrix K are considered. The matrix K here need not be square and is assumed to be a general real matrix of dimensions $r \times s$. Of course, K may also be a vector or scalar. Also, the attention of this section is restricted to differentiable functions and their gradient computations. In general the ARI's and other functions in the presented formulation for the \mathcal{H}_∞ - and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -problems depend on two parameter sets represented by the controller parameters in C_0 and the sought-after solutions to the inequality constraints. These two parameter sets are independent and hence gradients can be derived independently for the two sets. In this appendix the matrix K can either be the controller representation C_0 or the parameter matrix X^i . Assuming that a scalar function $f(K)$ is differentiable in all the elements of K , $f(K)$ can be linearized around a nominal point K_o as follows.

$$f(K_o + dK) = f(K_o) + df(K_o, dK) + r(dK). \quad (\text{A.31})$$

Here $df(K_o, dK)$ is linear in the variation dK of the parameter matrix. Furthermore $df(K_o, dK) = 0$ for $dK = 0$. The residual term $r(dK)$ collects all the higher-order terms in dK with $r(dK) = 0$ for $dK = 0$.

Gradient computations of scalar functions with respect to vectors have a long history in control theory. Many functions $f(K)$ can be converted to such a vector problem by using a vector representation $\text{vec}(K)$ of the individual elements of K , $K = K_{i,j}$ ($i = 1, 2, \dots, r$, $j = 1, 2, \dots, s$) as follows.

$$\text{vec}(K) = \begin{pmatrix} K_{1,1} & K_{2,1} & \dots & K_{r,1} & K_{1,2} & K_{2,2} & \dots & K_{r,2} & \dots & K_{r,s} \end{pmatrix}^T. \quad (\text{A.32})$$

Even if this is not possible, gradient expressions of differentiable scalar functions with respect to matrices can always be reduced to the vector case by forming a

corresponding vector representation $\text{vec}(dK)$ for the matrix dK to yield the expression

$$df(K_o, dK) = \left(\frac{\partial f}{\partial K_{1,1}} \quad \frac{\partial f}{\partial K_{1,2}} \quad \dots \quad \frac{\partial f}{\partial K_{r,s}} \right) \text{vec}(dK), \quad (\text{A.33})$$

which directly specifies the individual derivatives of $f(K)$ with respect to $K_{i,j}$. Accordingly we can define the same linearization for the matrix case. Consider a matrix-valued function $M(K)$ (not necessarily square) where every entry of $M(K)$ is differentiable in all individual matrix elements of K , then the matrix equivalent to (A.31) is

$$M(K_o + dK) = M(K_o) + dM(K_o, dK) + R(dK) \quad (\text{A.34})$$

with equivalent properties for $dM(K_o, dK)$ and $R(dK)$. A form of $dM(K_o, dK)$ corresponding to (A.33) can be derived by the application of the Kronecker product formula and a representation $\text{vec}(dK)$ of dK as in (A.33) (see [67]). However, for this work it is not necessary to invoke such tools since all the cost function gradients can be reduced to a form to which Kleinman's lemma is applicable. There is a multitude of results concerning gradients with respect to matrices for scalar functions, vector- and matrix-valued functions. A complete review of the underlying theory is beyond the scope of this appendix. In the following some results are reviewed that form a complete basis for the derivation of the cost function gradients necessary for the presented research and possible extensions as discussed in the body of this report. The reader is referred to [67] for a more in-depth information on this subject matter. The product rule and the Cauchy invariance theorem for nested functions are well known facts for scalar functions of one variable. Important for this research is that there are matrix equivalents for various scalar differentiation rules as follows (see [67]).

1. Linearity:

Let $G(K)$ and $H(K)$ be two matrix-valued functions of a real matrix K respectively, where $G(K)$ and $H(K)$ have compatible dimensions such that the matrix-valued function $M(K) = \alpha G(K) + \beta H(K)$ is defined for some real scalars α and β . Assume furthermore that both matrix-valued functions are differentiable at K_o , then

$$dM(K_o, dK) = \alpha dG(K_o, dK) + \beta dH(K_o, dK). \quad (\text{A.35})$$

2. Product rule for the matrix case:

Let $G(K)$ and $H(K)$ be two matrix-valued functions of a real matrix K respectively, where $G(K)$ and $H(K)$ have compatible dimensions such that $M(K) = G(K)H(K)$ is defined. Assume furthermore that both matrix-valued functions are differentiable at K_o , then

$$dM(K_o, dK) = [dG(K_o, dK)]H(K_o) + G(K_o)[dH(K_o, dK)]. \quad (\text{A.36})$$

3. Cauchy invariance theorem ([67], theorem 13, chapter 5, page 96):

Let $G(L)$ and $L = H(K)$ be two matrix-valued functions of real matrices L and K respectively, where $G(L)$ and $H(K)$ have compatible dimensions such that the nested function $M(K) = G[H(K)]$ is well defined. Assume furthermore that $G(L)$ is differentiable at L_o with $L_o = H(K_o)$ and $H(K)$ is differentiable at K_o , then

$$dM(K_o, dK) = dG[L_o, dH(K_o, dK)]. \quad (\text{A.37})$$

These important results will find repeated application in the computation of various cost function gradients derived in appendix B. In addition to the above theoretical framework a result is necessary that is related to matrix inverses. Assume that K is a quadratic matrix such that $|K| \neq 0$ for $K = K_o$ and consider the matrix-valued function $M(K) = K^{-1}$. For such a matrix function it has been shown ([67], theorem 3, page 151) that

$$dM(K_o, dK) = -K_o^{-1}(dK)K_o^{-1}. \quad (\text{A.38})$$

In general, with the product rule and the Cauchy's invariance theorem we can state a more general form for matrix functions involving matrix inverses. Consider a square matrix-valued function $M(K) = [G(K)H(K)]^{-1}$ with the assumption that $M(K)$ is nonsingular at K_o . Then it can be verified that

$$dM(K_o, dK) = -[M(K_o)]^{-1}[dG(K_o, dK)H(K_o) + G(K_o)dH(K_o, dK)][M(K_o)]^{-1}. \quad (\text{A.39})$$

These results are important and give rise to a whole range of explicit expressions for $dM(K_o, dK)$ for various matrix-valued functions $M(K)$. However, we are not interested in gradients of matrix expressions with respect to matrices. For the purposes of this research the goal is to find explicit closed-form gradients for scalar cost functions

with respect to matrices. All the above results are intermediate steps on the way to find such expressions for the defined cost functions. All gradient computations for these cost functions can, in one way or another, be transformed into forms that involve the trace functions. It is well known that the trace operator and the “ d -operator” are interchangeable, that is $d\text{Trace}[M(K_o, dK)] = \text{Trace}[dM(K_o, dK)]$. Hence the machinery developed above for matrix-valued functions will be applicable for this type of problem. For the derivation of explicit closed-form gradient expressions we then utilize the important lemma by Kleinman for this type of function.

Lemma A.2.1 (Kleinman’s Lemma, [133])

Consider the trace function $f(K) = \text{Trace}\{M(K)\}$ where $M(K)$ is a quadratic matrix function of a matrix $K \in R^{r \times s}$. Assume that $M(K)$ is (in all entries) differentiable with respect to every element of K . Assume furthermore, that $f(K_o + dK) - f(K_o)$ can be expressed as follows.

$$f(K_o + dK) - f(K_o) = \text{Trace}\{D(K_o)dK\}. \quad (\text{A.40})$$

Then the derivative of $f(K)$ at $K = K_o$ is given by

$$\left. \frac{\partial f(K)}{\partial K} \right|_{K=K_o} = [D(K_o)]^T \quad (\text{A.41})$$

where

$$\{[D(K_o)]^T\}_{k,l} = \frac{\partial f(K)}{\partial K_{k,l}}, \quad k = 1, 2, \dots, r, \quad l = 1, 2, \dots, s. \quad (\text{A.42})$$

A final remark to some notational convention used in this context. In many publications the following notational system has been used for matrix differentials.

$$\begin{aligned} M(K_o + \varepsilon \Delta K) &= M(K_o) + \Delta M(K_o, \varepsilon \Delta K) + \varepsilon^k R(\Delta K) \\ &= M(K_o) + \varepsilon \Delta M(K_o, \Delta K) + \varepsilon^k R(\Delta K) \end{aligned}$$

for $k > 1$. This notation is easily recovered from the one used in this thesis by applying the following substitutions.

$$\begin{aligned} dM(K_o, dK) &\rightarrow \varepsilon \Delta M(K_o, \Delta K) \\ dK &\rightarrow \Delta K. \end{aligned}$$

A.3 General Differential for $\text{Trace}\{e^{[M(K)]t_f}\}$

In this section a brief derivation of a general expression for the differential of the cost function $f(K, t_f) = \text{Trace}\{e^{[M(K)]t_f}\}$ in chapter 4 will be formalized and presented with some details. Assume that $M(K)$ is a real square symmetric matrix function of a parameter matrix K . Furthermore, $M(K)$ is assumed to be continuous and differentiable with respect to K . That is, we assume the existence of a $dM(K_o, dK)$ that satisfies (A.34). In the following let us assume furthermore, that t_f is given and that all elements of K are independent. Then we can consider $f(K, t_f)$ to be a function $g(K)$ of K only, namely $g(K) = f(K, t_f)$. Using a series expansion of the corresponding matrix exponential, $g(K)$ can then be expressed as follows.

$$\begin{aligned} g(K) &= \text{Trace}\{e^{[M(K)]t_f}\} \\ &= \text{Trace}\left\{\sum_{k=0}^{\infty} \frac{t_f^k}{k!} [M(K)]^k\right\} \\ &= \sum_{k=0}^{\infty} \frac{t_f^k}{k!} \text{Trace}\{[M(K)]^k\} \end{aligned} \quad (\text{A.43})$$

and hence by neglecting higher-order terms in dK and with $K = K_o + dK$,

$$g(K_o + dK) = \text{Trace}\left\{\sum_{k=0}^{\infty} \frac{t_f^k}{k!} [M(K_o + dK)]^k\right\} \quad (\text{A.44})$$

$$= \text{Trace}\left\{\sum_{k=0}^{\infty} \frac{t_f^k}{k!} [M(K_o) + dM(K_o, dK)]^k\right\} \quad (\text{A.45})$$

$$= \text{Trace}\left\{\sum_{k=0}^{\infty} \frac{t_f^k}{k!} [M(K_o)]^k + \sum_{k=1}^{\infty} \frac{t_f^k}{k!} k [M(K_o)]^{(k-1)} dM(K_o, dK)\right\} \quad (\text{A.46})$$

$$= g(K_o) + t_f \text{Trace}\left\{\sum_{k=1}^{\infty} \left[\frac{t_f^{(k-1)}}{(k-1)!} [M(K_o)]^{(k-1)}\right] dM(K_o, dK)\right\} \quad (\text{A.47})$$

$$= g(K_o) + t_f \text{Trace}\left\{\sum_{l=0}^{\infty} \left[\frac{t_f^l}{l!} (M(K_o))^l\right] dM(K_o, dK)\right\} \quad (\text{A.48})$$

$$= g(K_o) + t_f \text{Trace}\{e^{[M(K_o)]t_f} dM(K_o, dK)\}. \quad (\text{A.49})$$

Hence, for a given t_f , we arrive at the following result for this type of cost function.

$$dg(K_o, dK) = g(K_o + dK) - g(K_o) \quad (\text{A.50})$$

$$= t_f \text{Trace}\{e^{[M(K_o)]t_f} dM(K_o, dK)\} \quad (\text{A.51})$$

and hence for $f(K, t_f) = g(K)$:

$$df[(K_o, t_f), dK] = t_f \text{Trace}\{e^{[M(K_o)]t_f} dM(K_o, dK)\}. \quad (\text{A.52})$$

The only task left is the conversion of $dM(K_o, dK)$ into a form such that we can apply Kleinman's lemma to derive explicit gradients. This task, however, depends on the specific structure of $M(K)$ and results for some specific cost functions are derived in appendix B.

A.4 General Differentials of Eigenvalue Functions

Based on the fact that the above cost function $f(K, t_f)$ is expressible in terms of the individual eigenvalues $\lambda[M(K)]$ and for various other constraints a brief review of eigenvalue differentials is included here. Let $M(K)$ be a real square (not necessarily symmetric) matrix function of a real parameter matrix K . For this general case we restrict our attention to the case where the eigenvalues under consideration are simple and differentiable at $K = K_o$. Note that this is always true for a symmetric $M(K)$. Following the derivation in [67] let λ_o be an eigenvalue of $M(K_o)$. In general this eigenvalue will be a complex number $\lambda_o = \lambda_{or} + j\lambda_{oi}$ where the subscript r denotes the real part, the subscript i the imaginary part and $j = \sqrt{-1}$. Let u_o be the normalized right eigenvector of $M(K_o)$ associated with the eigenvalue λ_o and v_o the normalized right eigenvector of $M^T(K_o)$ associated with the eigenvalue $\lambda_{or} - j\lambda_{oi}$. These vectors are in general also complex vectors $u_o = u_{or} + ju_{oi}$ and $v_o = v_{or} + jv_{oi}$ respectively. Note that the definition of v_o implies that $(v_{or} - jv_{oi})^T$ is the normalized left eigenvector of $M(K_o)$ associated with the eigenvalue $\lambda_o = \lambda_{or} + j\lambda_{oi}$. We have by definition:

$$M(K_o)(u_{or} + ju_{oi}) = (\lambda_{or} + j\lambda_{oi})(u_{or} + ju_{oi}) \quad (\text{A.53})$$

$$M^T(K_o)(v_{or} + jv_{oi}) = (\lambda_{or} - j\lambda_{oi})(v_{or} + jv_{oi}) \quad (\text{A.54})$$

$$(u_{or} - ju_{oi})^T(u_{or} + ju_{oi}) = (v_{or} - jv_{oi})^T(v_{or} + jv_{oi}) = 1 \quad (\text{A.55})$$

for some eigenvalue λ_o of $M(K)$ at $K = K_o$. With these definitions the differential $d\lambda[M(K_o, dK)]$ can be expressed in the following form ([67], page 163).

$$d\lambda[M(K_o, dK)] = d\lambda_r[M(K_o, dK)] + jd\lambda_i[M(K_o, dK)] \quad (\text{A.56})$$

$$= \frac{(v_{or} - jv_{oi})^T dM(K_o, dK)(u_{or} + ju_{oi})}{(v_{or} - jv_{oi})^T (u_{or} + ju_{oi})} \quad (\text{A.57})$$

$$= \text{Trace}\left\{\frac{(u_{or} + ju_{oi})(v_{or} - jv_{oi})^T}{(v_{or} - jv_{oi})^T (u_{or} + ju_{oi})} dM(K_o, dK)\right\}. \quad (\text{A.58})$$

Stability constraints and symmetric matrix inequalities involve only the real part of all corresponding eigenvalues, the imaginary part λ_{oi} is irrelevant for such problems. Hence the differential $d\lambda_r[M(K_o, dK)]$ is required for this type of constraint or cost function, not $d\lambda[M(K_o, dK)]$. By combining equations (A.56) and (A.57) and examining the real and imaginary components separately (note $M(K)$ is a real matrix function of a real matrix and hence $dM(K_o, dK)$ is real), we arrive at the following linear system of equations

$$d\lambda_r[M(K_o, dK)]q_2 + d\lambda_i[M(K_o, dK)]q_1 = v_{or}^T dM(K_o, dK)u_{or} + v_{oi}^T dM(K_o, dK)u_{oi} \quad (\text{A.59})$$

$$d\lambda_r[M(K_o, dK)]q_1 + d\lambda_i[M(K_o, dK)]q_2 = v_{oi}^T dM(K_o, dK)u_{or} + v_{or}^T dM(K_o, dK)u_{oi} \quad (\text{A.60})$$

with the real scalars q_1 and q_2 given by

$$q_1 = v_{oi}^T u_{or} - v_{or}^T u_{oi} \quad (\text{A.61})$$

$$q_2 = v_{oi}^T u_{oi} + v_{or}^T u_{or}. \quad (\text{A.62})$$

The system (A.59) and (A.60) is readily solved for $d\lambda_r[M(K_o, dK)]$ to yield the following result for the differential of the real part of the eigenvalue.

$$d\lambda_r[M(K_o, dK)] = \frac{q_2}{q_1^2 + q_2^2} [v_{or}^T dM(K_o, dK)u_{or} + v_{oi}^T dM(K_o, dK)u_{oi}] + \frac{q_1}{q_1^2 + q_2^2} [v_{oi}^T dM(K_o, dK)u_{or} - v_{or}^T dM(K_o, dK)u_{oi}] \quad (\text{A.63})$$

$$= \text{Trace}\left\{\left[\frac{q_2}{q_1^2 + q_2^2}(u_{or}v_{or}^T + u_{oi}v_{oi}^T) + \frac{q_1}{q_1^2 + q_2^2}(u_{or}v_{oi}^T - u_{oi}v_{or}^T)\right]dM(K_o, dK)\right\} \quad (\text{A.64})$$

$$= \text{Trace}\{PdM(K_o, dK)\} \quad (\text{A.65})$$

with

$$P = \frac{q_2}{q_1^2 + q_2^2}(u_{or}v_{or}^T + u_{oi}v_{oi}^T) + \frac{q_1}{q_1^2 + q_2^2}(u_{or}v_{oi}^T - u_{oi}v_{or}^T). \quad (\text{A.66})$$

Note that $(q_1^2 + q_2^2)$ is non-zero (see [67]). Hence we have arrived once again at an expression that is amenable to the application of Kleinman's lemma once the structure of $M(K)$ is known. Of course, this formulation is valid only if the corresponding eigenvalue is simple. In the affirmative case we have derived a nice characterization of the differential in terms of the corresponding eigenvectors of $M(K_o)$ and $[M(K_o)]^T$ respectively.

The above expressions simplify considerably for the case when $M(K)$ is symmetric. In this case all eigenvalues are simple for any K . Moreover, all the eigenvalues and eigenvectors are real, and $u_o = v_o = u_S$. This implies that $q_1 = 0, q_2 = u_S^T u_S = 1$. Equation (A.65) is still valid for the symmetric case and we have

$$d\lambda_r[M(K_o, dK)] = d\lambda[M(K_o, dK)] = \text{Trace}\{u_S u_S^T dM(K_o, dK)\}. \quad (\text{A.67})$$

Although we have taken a different approach to arrive at this result for symmetric matrix functions, this result compares nicely to theorem 7 in [67], p. 159.

If the matrix-valued function $M(K)$ under consideration is not convex, then the cost function $\text{Trace}\{e^{[M(K)]t_f}\}$ is in general not convex either. This gives rise to a formulation for matrix inequality constraints $M(K) < 0$ in terms of the cost functions as defined in (4.13) or (4.15). In the following it will be shown, that gradients for these alternative cost functions can be found by minor modifications of the gradients for $\text{Trace}\{e^{[M(K)]t_f}\}$ without developing a whole new train of thought. Although this approach is applicable to general non-symmetric matrix functions, here only the symmetric case is presented. At this point consider a $q \times q$ symmetric matrix function $M(K)$ of a general real matrix K and the cost function $\hat{f}[M(K)]$.

$$\hat{f}[M(K)] = \sum_{k=1}^q \hat{f}_k[M(K)] \quad (\text{A.68})$$

$$\hat{f}_k[M(K)] = \begin{cases} t_f(\lambda_k[M(K)])^2 & \text{if } \lambda_k[M(K)] \geq 0 \\ 0 & \text{if } \lambda_k[M(K)] < 0. \end{cases} \quad (\text{A.69})$$

The corresponding differential is easily found to be

$$d\hat{f}[M(K_o), dK] = \sum_{k=1}^q d\hat{f}_k[M(K_o), dK] \quad (\text{A.70})$$

$$d\hat{f}_k[M(K_o), dK] = \begin{cases} 2t_f(\lambda_k[M(K_o)])d\lambda_k[M(K_o), dK] & \text{if } \lambda_k[M(K_o)] \geq 0 \\ 0 & \text{if } \lambda_k[M(K_o)] < 0 \end{cases} \quad (\text{A.71})$$

which, with (A.67), can be converted to

$$d\hat{f}[M(K_o), dK] = \sum_{k=1}^q d\hat{f}_k[M(K_o), dK] \quad (\text{A.72})$$

$$d\hat{f}_k[M(K_o), dK] = \begin{cases} 2t_f(\lambda_k[M(K_o)])\text{Trace}\{u_k u_k^T dM(K_o, dK)\} & \text{if } \lambda_k[M(K_o)] \geq 0 \\ 0 & \text{if } \lambda_k[M(K_o)] < 0 \end{cases} \quad (\text{A.73})$$

with

$$M(K_o)u_k = \lambda_k[M(K_o)]u_k, \quad \|u_k\|_2 = 1. \quad (\text{A.74})$$

Note at this point the structural equivalence between $d\hat{f}_k[M(K), dK]$ in (A.73) and $df[(K_o, t_f), dK]$ in (A.52). The only differences are the additional factor $2\lambda_k[M(K_o)]$ and the matrix $u_k u_k^T$ that substituted $e^{[M(K_o)]t_f}$ in $df[(K_o, t_f), dK]$. Hence, if the differential of $\text{Trace}\{e^{[M(K)]t_f}\}$ is known, the differential $d\hat{f}_k[M(K), dK]$ can be derived by the following scheme. If λ_k violates the desired inequality constraint, then

$$df[(K_o, t_f), dK] \rightarrow d\hat{f}_k[M(K_o), dK]$$

with the following substitutions in $df[(K_o, t_f), dK]$:

$$\begin{aligned} t_f &\rightarrow 2(\lambda_k[M(K_o)])t_f \\ e^{[M(K)]t_f} &\rightarrow u_k u_k^T \end{aligned}$$

to get the corresponding differential for the k^{th} eigenvalue of $M(K)$. The overall differential $d\hat{f}[M(K_o), dK]$ follows from the summation over the individual components. Hence the gradient computation for this type of cost function or for the cost functions in (4.13) can be derived directly from the gradients of the appropriate trace-function by simple substitution.

A.5 General Differentials of Functions Involving Grammians

Some design constraints involve controllability or observability grammians. In general these constraints will be of the form

$$f(K) = \text{Trace}\{e^{[M(K)-N]t_f}\} \quad (\text{A.75})$$

for some real constant symmetric matrix N and a real symmetric matrix function $M(K)$ that satisfies

$$A^T(K)M(K) + M(K)A(K) + Q(K) = 0. \quad (\text{A.76})$$

Note that $M(K)$ in this formulation does not depend explicitly on K , rather, its dependence on K stems from the facts that $A(K)$ and $Q(K)$ are matrix functions of K . In general it is assumed that $A(K)$ is a real square matrix function of K , that is continuous and differentiable with respect to K . Furthermore, $A(K)$ is restricted to be asymptotically stable at $K = K_o$ and $K = K_o + dK$. $Q(K)$ on the other hand is a real symmetric matrix function of K that is also continuous and differentiable in K . Due to continuity of $A(K)$ and $Q(K)$ and the monotonicity property of the eigenvalues of symmetric matrices (see Weyl's theorem) we can conclude the existence of an expression $dM(K_o, dK)$ such that

$$M(K_o + dK) = M(K_o) + dM(K_o, dK) + R(dK)$$

as in (A.34). Hence the machinery developed in equations (A.44) through (A.49) shows that

$$df(K_o, dK) = t_f \text{Trace}\{e^{[M(K_o)-N]t_f} dM(K_o, dK)\}. \quad (\text{A.77})$$

With the abbreviation $E = e^{[M(K_o)-N]t_f}$ this amounts to finding

$$df(K_o, dK) = t_f \text{Trace}\{E dM(K_o, dK)\}, \quad (\text{A.78})$$

subject to (A.76). With the assumption that $A(K)$ is asymptotically stable at K_o and $K = K_o + dK$, we can express $M(K)$ in terms of an integral over an infinite time-horizon.

$$M(K) = \lim_{t \rightarrow \infty} \int_0^t e^{[A(K)]^T \tau} Q(K) e^{A(K) \tau} d\tau. \quad (\text{A.79})$$

Due to the continuity and differentiability assumptions on $A(K)$ and $Q(K)$, corresponding expressions $dA(K_o, dK)$ and $dQ(K_o, dK)$ are also well defined. Using the fact [64] that

$$e^{[A(K_o) + dA(K_o, dK)]t} = e^{A(K_o)t} + \int_0^t e^{A(K_o)(t-s)} dA(K_o, dK) e^{A(K_o)s} ds \quad (\text{A.80})$$

we arrive (after some algebra) at the following differential expression for the function $f(K) = \text{Trace}\{e^{[M(K)-N]t_f}\}$.

$$df(K_o, dK) = t_f \text{Trace}\{L_1[2L_2 dA(K_o, dK) + dQ(K_o, dK)]\} \quad (\text{A.81})$$

where L_1 and L_2 solve the following Lyapunov equations,

$$A(K_o)L_1 + L_1A^T(K_o) + E = 0 \quad (\text{A.82})$$

$$L_2A(K_o) + A^T(K_o)L_2 + Q(K_o) = 0. \quad (\text{A.83})$$

For more details on this procedure the reader is referred to [64] and [133]. Without further details it is clear that similar expressions can be derived for the case where $M(K)$ solves

$$M(K)A^T(K) + A(K)M(K) + Q(K) = 0. \quad (\text{A.84})$$

In conclusion, for this type of function we also end up with an expression (A.81) to which we can apply Kleinman's lemma once the explicit structure of $A(K)$ and $Q(K)$ are known.

A.6 Gradients of Scalar Functions with Respect to Symmetric Matrices

So far only gradients of scalar functions $f(K)$ with respect to general matrices K have been considered. The case where K is symmetric needs some further elaboration. However, it can be shown ([90], chapter 10), that this special case can be reduced to the general case by use of theorem 10.1 in [90]. Under the usual continuity and differentiability conditions as above and the assumption that K enters $f(K)$ only in matrix form (that is $f(K)$ is not explicitly a function of individual elements of K), this theorem states, that for a symmetric matrix $K = K^T$ we have to modify the gradient expressions to account for this additional information as follows ([90]).

$$\frac{\partial f(K)}{\partial K} = \frac{\partial f(K)}{\partial K} + \frac{\partial f(K)}{\partial K^T} - \text{diag}\left\{\frac{\partial f(K)}{\partial K}\right\} \quad (\text{A.85})$$

where $\text{diag}\{\frac{\partial f(K)}{\partial K}\}$ has the same main diagonal as $\frac{\partial f(K)}{\partial K}$ and zero elements elsewhere. Note that gradient expression of a scalar cost function with respect to a symmetric matrix is itself symmetric. With (A.85) the gradient computation techniques developed above are also applicable to this problem with some extra effort due to the additional terms in the overall gradient.

Appendix B

DERIVATION OF EXPLICIT GRADIENT EXPRESSIONS

In this appendix we will provide a complete list of explicit closed-form gradient expressions for all the cost functions and constraint functions used to solve the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -control problem. The derivation and notation are based on the results presented in appendix A. For the general multi-plant case with output-feedback all gradients are based on a representation of the closed-loop systems in terms of the closed-loop system matrices defined in (3.17) which are repeated here for convenience.

$$\begin{aligned}
A_{cl,2}^i &= \bar{A}^i + \bar{B}_3^i C_o \bar{C}_3^i \\
B_{cl,2}^i &= \bar{B}_1^i + \bar{B}_3^i C_o \bar{D}_{31}^i \\
C_{cl,2}^i &= \bar{C}_1^i + \bar{D}_{13}^i C_o \bar{C}_3^i \\
A_{cl,\infty}^i &= A_{cl,2}^i = A_{cl}^i \\
B_{cl,\infty}^i &= \bar{B}_2^i + \bar{B}_3^i C_o \bar{D}_{32}^i \\
C_{cl,\infty}^i &= \bar{C}_2^i + \bar{D}_{23}^i C_o \bar{C}_3^i \\
D_{cl,\infty}^i &= \bar{D}_{22}^i + \bar{D}_{23}^i C_o \bar{D}_{32}^i,
\end{aligned}$$

and

$$\begin{aligned}
R^i &= (\gamma^i)^2 I - (D_{cl,\infty}^i)^T D_{cl,\infty}^i \\
S^i &= (\gamma^i)^2 I - D_{cl,\infty}^i (D_{cl,\infty}^i)^T.
\end{aligned}$$

The individual matrices \bar{A}^i , \bar{B}_3^i and so forth are defined in chapter 3. In the following gradient expressions are derived for the i^{th} plant condition only. Depending on the formulation of the actual optimization problem, gradient expressions for all n_p plant conditions can be formed from these individual gradient expressions. Furthermore, the subscript “ $(*)_o$ ” has been used in appendix A to denote the point at which the gradient is evaluated. This convention is dropped here in order to simplify the notation. Hence in the following C_0 and X^i are used instead of C_{0_o} and $X_{o_o}^i$ respectively for simplicity.

B.1 \mathcal{H}_∞ -Constraint Function Gradients – General Case

In this section we consider the cost functions associated with the pure \mathcal{H}_∞ -problem and the corresponding cost/constraint functions in (5.1.1). The first cost function is

$$f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i) = \text{Trace} \{ e^{ARI_{C,OF}^i(C_0, X^i, \gamma^i) t_{f1}^i} \} \quad (\text{B.1})$$

associated with the i^{th} plant condition. In the following, various intermediate steps are shown for the computation of the ARI-gradients. Because the intermediate steps require quite a complex notation, these details will be omitted for the other cost function gradients which can be derived using the same techniques. Two forms of $ARI_{C,OF}^i(C_0, X^i, \gamma^i)$ will be used in deriving the gradients with respect to C_0 and X^i . They are the two equivalent forms of $ARI_{C,OF}^i(C_0, X^i, \gamma^i)$ given in (2.31) and (2.30). The formulation

$$ARI_{C,OF}^i(C_0, X^i, \gamma^i) = (A_{cl}^i)^T X^i + X^i A_{cl}^i + (C_{cl,\infty}^i)^T C_{cl,\infty}^i + P_{aux}^i (R^i)^{-1} (P_{aux}^i)^T \quad (\text{B.2})$$

with

$$P_{aux}^i = X^i B_{cl,\infty}^i + (C_{cl,\infty}^i)^T D_{cl,\infty}^i \quad (\text{B.3})$$

provides a convenient form of $ARI_{C,OF}^i(C_0, X^i, \gamma^i)$ for the gradient computation with respect to C_0 while

$$ARI_{C,OF}^i(C_0, X^i, \gamma^i) = (A_{aux}^i)^T X^i + X^i A_{aux}^i + X^i B_{aux}^i X^i + C_{aux}^i \quad (\text{B.4})$$

with

$$A_{aux}^i = A_{cl}^i + B_{cl,\infty}^i (R^i)^{-1} (D_{cl,\infty}^i)^T C_{cl,\infty}^i \quad (\text{B.5})$$

$$B_{aux}^i = B_{cl,\infty}^i (R^i)^{-1} (B_{cl,\infty}^i)^T \quad (\text{B.6})$$

$$C_{aux}^i = (\gamma^i)^2 (C_{cl,\infty}^i)^T (S^i)^{-1} C_{cl,\infty}^i \quad (\text{B.7})$$

is more appropriate for the gradients with respect to X^i . The gradients of (B.1) with respect to the matrix C_0 will be derived first. Note that for this derivation X^i is treated as a constant. With the abbreviation

$$E_{ARI_{C,OF}}^i = e^{ARI_{C,OF}^i(C_0, X^i, \gamma^i) t_{f1}^i} = (E_{ARI_{C,OF}}^i)^T \quad (\text{B.8})$$

the result in (A.52) can be applied to the cost function (B.1) with $ARI_{C,OF}^i(C_0, X^i, \gamma^i)$ as given in (B.2) to yield

$$\begin{aligned}
& df_{ARI_{C,OF}}^i[(C_0, X^i, \gamma^i, t_{f1}^i), dC_0] \\
&= t_{f1}^i \text{Trace}\{E_{ARI_{C,OF}}^i dARI_{C,OF}^i[(C_0, X^i, \gamma^i), dC_0]\} \\
&= t_{f1}^i \text{Trace}\{E_{ARI_{C,OF}}^i((\bar{C}_3^i)^T(dC_0)^T(\bar{B}_3^i)^T X^i + X^i \bar{B}_3^i(dC_0)^T \bar{C}_3^i \\
&\quad + (\bar{C}_3^i)^T(dC_0)^T(\bar{D}_{23}^i)^T C_{cl,\infty}^i + (C_{cl,\infty}^i)^T \bar{D}_{23}^i(dC_0)^T \bar{C}_3^i \\
&\quad + [X^i \bar{B}_3^i(dC_0)^T \bar{D}_{32}^i + (\bar{C}_3^i)^T(dC_0)^T(\bar{D}_{23}^i)^T + (C_{cl,\infty}^i)^T \bar{D}_{23}^i(dC_0)^T \bar{C}_3^i](R^i)^{-1}(P_{aux}^i)^T \\
&\quad + P_{aux}^i(R^i)^{-1}[X^i \bar{B}_3^i(dC_0)^T \bar{D}_{32}^i + (\bar{C}_3^i)^T(dC_0)^T(\bar{D}_{23}^i)^T + (C_{cl,\infty}^i)^T \bar{D}_{23}^i(dC_0)^T \bar{C}_3^i]^T \\
&\quad + P_{aux}^i(R^i)^{-1}[(\bar{D}_{32}^i)^T(dC_0)^T(\bar{D}_{23}^i)^T D_{cl,\infty}^i + (D_{cl,\infty}^i)^T \bar{D}_{23}^i(dC_0)^T \bar{D}_{32}^i](R^i)^{-1}(P_{aux}^i)^T)\}.
\end{aligned} \tag{B.9}$$

For individual steps the reader is referred to the machinery developed in appendix A. In particular the linearity property, the product rule for matrix differentials and the explicit differential expressions for functions involving matrix inverses as given in (A.38) and (A.39) have been utilized to derive (B.9). After some simplifications and rearrangement we arrive at the more compact form

$$\begin{aligned}
& df_{ARI_{C,OF}}^i[(C_0, X^i, \gamma^i, t_{f1}^i), dC_0] \\
&= 2t_{f1}^i \text{Trace}\{[\bar{C}_3^i + \bar{D}_{32}^i(R^i)^{-1}(P_{aux}^i)^T]E_{ARI_{C,OF}}^i \\
&\quad [X^i \bar{B}_3^i + ((C_{cl,\infty}^i)^T + P_{aux}^i(R^i)^{-1}(D_{cl,\infty}^i)^T)\bar{D}_{23}^i]dC_0\}
\end{aligned} \tag{B.10}$$

and hence by applying Kleinman's lemma to this trace function the gradient of $f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)$ with respect to C_0 is given by

$$\begin{aligned}
\frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial C_0} &= 2t_{f1}^i \{[\bar{C}_3^i + \bar{D}_{32}^i(R^i)^{-1}(P_{aux}^i)^T]E_{ARI_{C,OF}}^i \\
&\quad [X^i \bar{B}_3^i + ((C_{cl,\infty}^i)^T + P_{aux}^i(R^i)^{-1}(D_{cl,\infty}^i)^T)\bar{D}_{23}^i]\}^T
\end{aligned} \tag{B.11}$$

for the i^{th} plant condition. Note that the computation of this expression involves only elementary matrix operations such as matrix multiplication, matrix addition and one matrix exponential for $E_{ARI_{C,OF}}^i$ at each plant condition. Next the gradients of (B.1) with respect to X^i are computed. At this point no symmetry assumption on X^i is imposed. Later on it is shown how this symmetry constraint may be accounted for

explicitly as pointed out in chapter 5.

$$\begin{aligned}
df_{ARI_{C,OF}}^i[(C_0, X^i, \gamma^i, t_{f1}^i), dX^i] &= t_{f1}^i \text{Trace}\{E_{ARI_{C,OF}}^i dARI_{C,OF}^i[(C_0, X^i, \gamma^i), dX^i]\} \\
&= t_{f1}^i \text{Trace}\{E_{ARI_{C,OF}}^i[(A_{aux}^i)^T dX^i + dX^i A_{aux}^i + dX^i B_{aux}^i X^i + X^i B_{aux}^i dX^i]\} \\
&= t_{f1}^i \text{Trace}\{E_{ARI_{C,OF}}^i(A_{aux}^i + B_{aux}^i X^i)^T dX^i + dX^i(A_{aux}^i + B_{aux}^i X^i)^T E_{ARI_{C,OF}}^i\} \\
&= t_{f1}^i \text{Trace}\{[E_{ARI_{C,OF}}^i(A_{aux}^i + B_{aux}^i X^i)^T + (A_{aux}^i + B_{aux}^i X^i)E_{ARI_{C,OF}}^i]dX^i\}.
\end{aligned} \tag{B.12}$$

Applying Kleinman's lemma to this expression directly gives the desired gradient expression in a closed form.

$$\begin{aligned}
\frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial X^i} &= t_{f1}^i \{E_{ARI_{C,OF}}^i(A_{aux}^i + B_{aux}^i X^i)^T \\
&\quad + (A_{aux}^i + B_{aux}^i X^i)E_{ARI_{C,OF}}^i\}.
\end{aligned} \tag{B.13}$$

Finally, for the problem of designing a \mathcal{H}_∞ -optimal controller, gradients of the cost function $f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)$ with respect to γ^i are required. Note that $ARI_{C,OF}^i(C_0, X^i, \gamma^i)$ only contains terms in $(\gamma^i)^2$. Hence, for this problem it is sufficient to optimize over $(\gamma^i)^2$. Without further details and assuming all the above abbreviations it can be verified that

$$\frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial (\gamma^i)^2} = -t_{f1}^i \text{Trace}\{(R^i)^{-1}(P_{aux}^i)^T E_{ARI_{C,OF}}^i P_{aux}^i (R^i)^{-1}\}. \tag{B.14}$$

Differentials for the remaining \mathcal{H}_∞ -related constraint functions $f_D^i(C_0, \gamma^i, t_{f2}^i)$ and $f_X^i(X^i, t_{f3}^i)$ in definition 5.1.1 are treated next. Recall that

$$f_D^i(C_0, \gamma^i, t_{f2}^i) = \text{Trace}\{e^{[D_{cl,\infty}^{iT} D_{cl,\infty}^i - (\gamma^i)^2 I]t_{f2}^i}\} \tag{B.15}$$

$$f_X^i(X^i, t_{f3}^i) = \text{Trace}\{e^{-X^i t_{f3}^i}\}, \tag{B.16}$$

and hence the corresponding differentials are as follows.

$$df_D^i[(C_0, \gamma^i, t_{f2}^i), dC_0] = 2t_{f2}^i \text{Trace}\{\bar{D}_{32} E_D^i (D_{cl,\infty})^T \bar{D}_{23} dC_0\} \tag{B.17}$$

$$df_D^i[(C_0, \gamma^i, t_{f2}^i), dX^i] = 0 \tag{B.18}$$

$$df_D^i[(C_0, \gamma^i, t_{f2}^i), d(\gamma^i)^2] = -t_{f2}^i \text{Trace}\{E_D^i d(\gamma^i)^2\} \tag{B.19}$$

$$df_X^i[X^i, t_{f3}^i], dC_0] = 0 \tag{B.20}$$

$$df_X^i[X^i, t_{f3}^i], dX^i] = -t_{f3}^i \text{Trace}\{E_X^i dX^i\} \tag{B.21}$$

$$df_X^i[X^i, t_{f3}^i], d(\gamma^i)^2] = 0 \tag{B.22}$$

where

$$E_D^i = e^{[D_{cl,\infty}^{iT} D_{cl,\infty}^i - (\gamma^i)^2] t_{f2}^i} \quad (\text{B.23})$$

$$E_X^i = e^{-X^i t_{f3}^i}. \quad (\text{B.24})$$

Kleinman's lemma immediately gives the corresponding gradient expressions. For easier reference these gradient expressions and the above gradients for the ARI-constraint functions are summarized below.

$$1. \quad \underline{f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i) = \text{Trace} \{ e^{ARI_{C,OF}^i(C_0, X^i, \gamma^i) t_{f1}^i} \} :}$$

$$\begin{aligned} \frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial C_0} &= 2t_{f1}^i \{ [\bar{C}_3^i + \bar{D}_{32}^i (R^i)^{-1} (P_{aux}^i)^T] E_{ARI_{C,OF}}^i \\ &\quad [X^i \bar{B}_3^i + ((C_{cl,\infty}^i)^T + P_{aux}^i (R^i)^{-1} (D_{cl,\infty}^i)^T) \bar{D}_{23}^i] \}^T \end{aligned} \quad (\text{B.25})$$

$$\begin{aligned} \frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial X^i} &= t_{f1}^i \{ E_{ARI_{C,OF}}^i (A_{aux}^i + B_{aux}^i X^i)^T \\ &\quad + (A_{aux}^i + B_{aux}^i X^i) E_{ARI_{C,OF}}^i \} \end{aligned} \quad (\text{B.26})$$

$$\begin{aligned} \frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial (\gamma^i)^2} &= -t_{f1}^i \text{Trace} \{ (R^i)^{-1} (P_{aux}^i)^T E_{ARI_{C,OF}}^i P_{aux}^i (R^i)^{-1} \} \end{aligned} \quad (\text{B.27})$$

with

$$\begin{aligned} ARI_{C,OF}^i(C_0, X^i, \gamma^i) &= (A_{aux}^i)^T X^i + X^i A_{aux}^i + X^i B_{aux}^i X^i + C_{aux}^i \\ R^i &= (\gamma^i)^2 I - (D_{cl,\infty}^i)^T D_{cl,\infty}^i \\ S^i &= (\gamma^i)^2 I - D_{cl,\infty}^i (D_{cl,\infty}^i)^T \\ A_{aux}^i &= A_{cl}^i + B_{cl,\infty}^i (R^i)^{-1} (D_{cl,\infty}^i)^T C_{cl,\infty}^i \\ B_{aux}^i &= B_{cl,\infty}^i (R^i)^{-1} (B_{cl,\infty}^i)^T \\ C_{aux}^i &= (\gamma^i)^2 (C_{cl,\infty}^i)^T (S^i)^{-1} C_{cl,\infty}^i \\ P_{aux}^i &= X^i B_{cl,\infty}^i + (C_{cl,\infty}^i)^T D_{cl,\infty}^i \\ E_{ARI_{C,OF}}^i &= e^{ARI_{C,OF}^i(C_0, X^i, \gamma^i) t_{f1}^i} \\ i &= 1, 2, \dots, n_p. \end{aligned}$$

2. $f_D^i(C_0, \gamma^i, t_{f2}^i) = \text{Trace} \{ e^{[D_{cl,\infty}^{iT} D_{cl,\infty}^i - (\gamma^i)^2 \eta] t_{f2}^i} \}$:

$$\frac{\partial f_D^i(C_0, \gamma^i, t_{f2}^i)}{\partial C_0} = 2t_{f2}^i [\bar{D}_{32} E_D^i (D_{cl,\infty})^T \bar{D}_{23}]^T \quad (\text{B.28})$$

$$\frac{\partial f_D^i(C_0, \gamma^i, t_{f2}^i)}{\partial X^i} = 0 \quad (\text{B.29})$$

$$\frac{\partial f_D^i(C_0, \gamma^i, t_{f2}^i)}{\partial (\gamma^i)^2} = -t_{f2}^i \text{Trace} \{ E_D^i \} \quad (\text{B.30})$$

with

$$\begin{aligned} E_D^i &= e^{[D_{cl,\infty}^{iT} D_{cl,\infty}^i - (\gamma^i)^2 \eta] t_{f2}^i} \\ i &= 1, 2, \dots, n_p. \end{aligned}$$

3. $f_X^i(X^i, t_{f3}^i) = \text{Trace} \{ e^{-X^i t_{f3}^i} \}$:

$$\frac{\partial f_X^i(X^i, t_{f3}^i)}{\partial C_0} = 0 \quad (\text{B.31})$$

$$\frac{\partial f_X^i(X^i, t_{f3}^i)}{\partial X^i} = -t_{f3}^i E_X^i \quad (\text{B.32})$$

$$\frac{\partial f_X^i(X^i, t_{f3}^i)}{\partial (\gamma^i)^2} = 0 \quad (\text{B.33})$$

with

$$\begin{aligned} E_X^i &= e^{-X^i t_{f3}^i} \\ i &= 1, 2, \dots, n_p. \end{aligned}$$

B.1.1 Modification of the Gradient Expressions for Symmetric X^i

Note that for the gradient expressions as derived in the previous sections, the fact that all X^i are symmetric has not been used. The discussion in chapter 5 shows that

the optimization can be performed over a set of n_p upper triangular matrices \tilde{X}^i

$$\tilde{X}^i = \begin{pmatrix} \tilde{X}_{1,1}^i & \tilde{X}_{1,2}^i & \tilde{X}_{1,3}^i & \cdots & \tilde{X}_{1,(n_{x^i}+n_c-1)}^i & \tilde{X}_{1,(n_{x^i}+n_c)}^i \\ 0 & \tilde{X}_{2,2}^i & \tilde{X}_{2,3}^i & \cdots & \tilde{X}_{2,(n_{x^i}+n_c-1)}^i & \tilde{X}_{2,(n_{x^i}+n_c)}^i \\ 0 & 0 & \tilde{X}_{3,3}^i & \cdots & \tilde{X}_{3,(n_{x^i}+n_c-1)}^i & \tilde{X}_{3,(n_{x^i}+n_c)}^i \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \tilde{X}_{(n_{x^i}+n_c-1),(n_{x^i}+n_c-1)}^i & \tilde{X}_{(n_{x^i}+n_c-1),(n_{x^i}+n_c)}^i \\ 0 & 0 & 0 & \cdots & 0 & \tilde{X}_{(n_{x^i}+n_c),(n_{x^i}+n_c)}^i \end{pmatrix} \quad (\text{B.34})$$

from which the matrices X^i are formed by $X^i = \tilde{X}^i + (\tilde{X}^i)^T - \text{diag}[\tilde{X}^i]$ or $X^i = (\tilde{X}^i)^T \tilde{X}^i$. In the following discussion the function $f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)$ is used as an example to illustrate the necessary modifications to account for one of these factorizations. These modifications are easily applied to the other relevant functions and their gradients. Also, note that the gradient expressions for C_0 are not affected by a particular form of X_i and remain unchanged.

Let us first assume a factorization $X^i = \tilde{X}^i + (\tilde{X}^i)^T - \text{diag}[\tilde{X}^i]$. The cost function becomes

$$f_{ARI_{C,OF}}^i(C_0, \tilde{X}^i, \gamma^i, t_{f1}^i) = f_{ARI_{C,OF}}^i(C_0, X^i = \tilde{X}^i + (\tilde{X}^i)^T - \text{diag}[\tilde{X}^i], \gamma^i, t_{f1}^i)$$

Now gradients with respect to \tilde{X}^i have to be computed. Following the derivation of $df_{ARI_{C,OF}}^i[(C_0, X^i, \gamma^i, t_{f1}^i), dX^i]$ in (B.12) it can be shown that

$$\frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial X^i} = \frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial (X^i)^T} \quad (\text{B.35})$$

and hence, by invoking (A.85), the gradients of $f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)$ with respect to a symmetric X^i amount to

$$\begin{aligned} \frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial X^i} \Big|_{X^i=(X^i)^T} &= 2 \frac{\partial f_{ARI_{C,OF}}^i(C_0, \tilde{X}^i, \gamma^i, t_{f1}^i)}{\partial \tilde{X}^i} \\ &\quad - \text{diag} \left\{ \frac{\partial f_{ARI_{C,OF}}^i(C_0, \tilde{X}^i, \gamma^i, t_{f1}^i)}{\partial \tilde{X}^i} \right\}. \end{aligned} \quad (\text{B.36})$$

This expression gives rise to the following gradient expressions with respect to \tilde{X}^i :

$$\frac{\partial f_{ARI_{C,OF}}^i(C_0, \tilde{X}^i, \gamma^i, t_{f1}^i)}{\partial \tilde{X}_{k,l}^i} = \left[\frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial X^i} \Big|_{X^i=(X^i)^T} \right]_{k,l} \quad (\text{B.37})$$

for $k = 1, 2, \dots, (n_{x^i} + n_c)$, $l = k + m$, $m = 0, 1, \dots, n_{x^i} + n_c - k$. However, due to the fact that $\frac{\partial f_{ARI_{C,OF}}^{(C_0, X^i, \gamma^i, t_{f1}^i)}(\gamma^i)}{\partial X^i}$ is a symmetric matrix, an equivalent expression for these gradients can be defined in terms of (B.13) as follows.

$$\frac{\partial f_{ARI_{C,OF}}^{(C_0, \tilde{X}^i, \gamma^i, t_{f1}^i)}}{\partial \tilde{X}_{k,l}^i} = \begin{cases} \left[\frac{\partial f_{ARI_{C,OF}}^{(C_0, X^i, \gamma^i, t_{f1}^i)}}{\partial X^i} \right]_{k,l} & \text{if } k = l \\ 2 \left[\frac{\partial f_{ARI_{C,OF}}^{(C_0, X^i, \gamma^i, t_{f1}^i)}}{\partial X^i} \right]_{k,l} & \text{if } k < l \end{cases} \quad (\text{B.38})$$

for $k = 1, 2, \dots, (n_{x^i} + n_c)$, $l = k + m$, $m = 0, 1, \dots, n_{x^i} + n_c - k$. Note that such an equivalence may not be possible if $\frac{\partial f_{ARI_{C,OF}}^{(C_0, X^i, \gamma^i, t_{f1}^i)}}{\partial X^i}$ is not a symmetric matrix! However, symmetry does hold for all the cost functions and their gradients with respect to X^i . Hence (B.38) is equally applicable to all the other functions considered here.

Numerically both expressions are equivalent. Computationally the formulation in (B.38) is more effective since it requires less matrix operations than that in (B.37). Mathematically, however, (B.36) and hence (B.37) represent the correct gradient expressions for the considered factorization.

Alternatively, symmetry of X^i can be imposed by optimizing over the Cholesky factor \tilde{X}^i of $X^i = (\tilde{X}^i)^T \tilde{X}^i$. Note that the desired gradients of

$$f_{ARI_{C,OF}}^i(C_0, X^i = (\tilde{X}^i)^T \tilde{X}^i, \gamma^i, t_{f1}^i)$$

with respect to \tilde{X}^i do not require the special treatment for symmetric matrices as above. \tilde{X}^i can be considered a general square matrix and the gradient expression will contain the gradients with respect to all the elements of \tilde{X}^i . Of all of these individual gradients, the gradient-based optimization utilizes only these gradients that correspond to a non-zero element in \tilde{X}^i . For this parametrization the partial gradients with respect to \tilde{X}^i are easily derived from (B.13). With Kleinman's lemma

and equation (B.13) the following expressions can be derived.

$$\begin{aligned}
df_{ARI_{C,OF}}^i & [(C_0, X^i = (\tilde{X}^i)^T \tilde{X}^i, \gamma^i, t_{f1}^i), d\tilde{X}^i] \\
&= t_{f1}^i \text{Trace} \left\{ \left[\frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial X^i} \right]^T [(d\tilde{X}^i)^T \tilde{X}^i + (\tilde{X}^i)^T d\tilde{X}^i] \right\} \\
&= 2t_{f1}^i \text{Trace} \left\{ \left[\frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial X^i} \right]^T (\tilde{X}^i)^T d\tilde{X}^i \right\}
\end{aligned} \tag{B.39}$$

Hence, by application of Kleinman's lemma,

$$\frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i = (\tilde{X}^i)^T \tilde{X}^i, \gamma^i, t_{f1}^i)}{\partial \tilde{X}^i} = 2\tilde{X}^i \frac{\partial f_{ARI_{C,OF}}^i(C_0, X^i, \gamma^i, t_{f1}^i)}{\partial X^i}. \tag{B.40}$$

Thus the gradients with respect to \tilde{X}^i differ from the gradients with respect to X^i only by the multiplicative factor $2\tilde{X}^i$. This is generally true for cost functions that depend on X^i with \tilde{X}^i as the optimization variable and $X^i = (\tilde{X}^i)^T \tilde{X}^i$. In either case the gradients with respect to \tilde{X}^i can be derived from a modification of the gradients with respect to X^i .

B.2 \mathcal{H}_2 -Performance Cost Gradients – General Case

This section is devoted to the gradient computation of the \mathcal{H}_2 -performance cost functional $J_2(C_0, t_{f\mathcal{H}_2})$ defined in definition 3.2.1. Gradients for the most general finite-time case have been derived in [64] and are not repeated here. The gradients as presented here require an internally stabilizing controller C_0 such that the n_p closed-loop system matrices A_{cl}^i are stable. Thus the limiting case of $t_{f\mathcal{H}_2} \rightarrow \infty$ can be used to derive the necessary gradients. As in the last section attention is restricted only to the i^{th} plant condition. The overall gradients can be obtained from the summation over all the plant conditions. Note that this cost function is independent of the parameters X^i and γ^i . Hence only gradients with respect to C_0 have to be computed. For this case the i^{th} cost function is

$$J_2^i(C_0, \infty) = \lim_{t_{f\mathcal{H}_2} \rightarrow \infty} J_2^i(C_0, t_{f\mathcal{H}_2}) \tag{B.41}$$

$$= \lim_{t_{f\mathcal{H}_2} \rightarrow \infty} \mathcal{E}[z_2^{iT}(t_{f\mathcal{H}_2}) z_2^i(t_{f\mathcal{H}_2})] \tag{B.42}$$

$$= \lim_{t_{f\mathcal{H}_2} \rightarrow \infty} \text{Trace} \left\{ \int_0^{t_{f\mathcal{H}_2}} (B_{cl,2}^i)^T e^{(A_{cl}^i)^T t} (C_{cl,2}^i)^T C_{cl,2}^i e^{A_{cl}^i t} B_{cl,2}^i dt \right\} \quad (\text{B.43})$$

$$= \text{Trace} \{ (B_{cl,2}^i)^T L_o^i B_{cl,2}^i dt \} \quad (\text{B.44})$$

$$= \text{Trace} \{ B_{cl,2}^i (B_{cl,2}^i)^T L_o^i \} \quad (\text{B.45})$$

where L_o solves

$$(A_{cl}^i)^T L_o^i + L_o^i A_{cl}^i + (C_{cl,2}^i)^T C_{cl,2}^i = 0. \quad (\text{B.46})$$

Applying equations (A.79) through (A.83) in appendix A we obtain the following differential expressions.

$$dJ_2^i[(C_0, \infty), dC_0] = 2\text{Trace} \{ L_1^i [L_2^i \bar{B}_3^i (dC_0) \bar{C}_3^i + (C_{cl,2}^i)^T \bar{D}_{23}^i (dC_0) \bar{C}_3^i] \} \quad (\text{B.47})$$

$$= 2\text{Trace} \{ \bar{C}_3^i L_1^i [L_2^i \bar{B}_3^i + (C_{cl,2}^i)^T \bar{D}_{23}^i] dC_0 \} \quad (\text{B.48})$$

with

$$A_{cl}^i L_1^i + L_1^i (A_{cl}^i)^T + B_{cl,2}^i (B_{cl,2}^i)^T = 0 \quad (\text{B.49})$$

$$L_2^i A_{cl}^i + (A_{cl}^i)^T L_2^i + (C_{cl,2}^i)^T C_{cl,2}^i = 0. \quad (\text{B.50})$$

It follows that the gradient expression is given by

$$\frac{\partial J_2^i(C_0, \infty)}{\partial C_0} = 2 \{ \bar{C}_3^i L_1^i [L_2^i \bar{B}_3^i + (C_{cl,2}^i)^T \bar{D}_{23}^i] \}^T. \quad (\text{B.51})$$

For more details on the derivation the reader is referred to [64].

B.3 Gradients for the Full State-Feedback Case: Continuous-Time Domain

This section contains gradients for the state-feedback case as considered in chapter 6. With the controller factorization $C_0 = W X^{-1}$ and the abbreviations given in chapter 6 we need to find the partial gradients of

$$J_{2,SF}(W, X) = \text{Trace} \{ C_{cl,2} X C_{cl,2}^T \} \quad (\text{B.52})$$

$$= \text{Trace} \{ [C_1 + D_{13} W X^{-1}] X [C_1 + D_{13} W X^{-1}]^T \} \quad (\text{B.53})$$

and

$$J_{\infty,SF}(W, X, \gamma, t_{f1}, t_{f3}) = \text{Trace} \{ e^{A_{RI_{C,SF}}(W,X,\gamma)t_{f1}} + e^{-X t_{f3}} \} \quad (\text{B.54})$$

where

$$\begin{aligned} ARI_{C,SF}(W, X, \gamma) &:= X[A + B_3 W X^{-1}]^T + [A + B_3 W X^{-1}]X \\ &\quad + \gamma^{-2} X[C_2 + D_{23} W X^{-1}]^T [C_2 + D_{23} W X^{-1}]X + B_1 B_1^T \end{aligned} \quad (B.55)$$

with respect to W and X . Using the tools in appendix A, the following differential can be derived.

$$\begin{aligned} dJ_{2,SF}[(W, X), dW] &= \text{Trace}\{D_{13}(dW)X^{-1}XC_{cl,2}^T + C_{cl,2}XX^{-1}(dW)^T D_{13}^T\} \\ &= 2\text{Trace}\{C_{cl,2}^T D_{13}(dW)\}, \end{aligned} \quad (B.56)$$

and hence, by applying Kleinman's lemma, we obtain

$$\frac{\partial J_{2,SF}(W, X)}{\partial W} = 2D_{13}^T C_{cl,2}. \quad (B.57)$$

Equivalently, for the partial gradients with respect to X we have

$$\begin{aligned} dJ_{2,SF}[(W, X), dX] &= \text{Trace}\{-D_{13}W X^{-1}(dX)X^{-1}XC_{cl,2}^T + C_{cl,2}(dX)C_{cl,2}^T \\ &\quad - C_{cl,2}XX^{-1}(dX)X^{-1}XW^T D_{13}^T\} \end{aligned} \quad (B.58)$$

$$= \text{Trace}\{[C_1^T C_1 - X^{-1}W^T D_{13}^T D_{13}W X^{-1}](dX)\} \quad (B.59)$$

and

$$\frac{\partial J_{2,SF}(W, X)}{\partial X} = C_1^T C_1 - X^{-1}W^T D_{13}^T D_{13}W X^{-1}. \quad (B.60)$$

As $J_{2,SF}(W, X)$ is not a function of γ , the corresponding partial gradient is zero. Now consider the cost function $J_{\infty,SF}(W, X, \gamma, t_{f1}, t_{f3})$ for fixed t_{f1} and t_{f3} . Applying the same machinery as before to the problem at hand, the following gradient expressions can be derived.

$$\frac{\partial J_{\infty,SF}(W, X, \gamma, t_{f1}, t_{f3})}{\partial W} = 2t_{f1}[B_3^T + \gamma^{-2}D_{23}^T(C_2 X + D_{23}W)]E_{ARI_{C,SF}} \quad (B.61)$$

$$\begin{aligned} \frac{\partial J_{\infty,SF}(W, X, \gamma, t_{f1}, t_{f3})}{\partial X} &= t_{f1}\{[A + \gamma^{-2}(C_2 X + D_{23}W)^T C_2]E_{ARI_{C,SF}} + \\ &\quad E_{ARI_{C,SF}}[A + \gamma^{-2}(C_2 X + D_{23}W)^T C_2]^T\} - t_{f3}e^{-X t_{f3}} \end{aligned} \quad (B.62)$$

$$\frac{\partial J_{\infty,SF}(W, X, \gamma, t_{f1}, t_{f3})}{\partial \gamma^2} = -t_{f1} \gamma^{-4} \text{Trace}\{E_{ARI_{C,SF}} X C_{cl,\infty}^T C_{cl,\infty} X\} \quad (B.63)$$

where

$$E_{ARI_{C,SF}} = e^{ARI_{C,SF}(W,X,\gamma)t_{f1}}. \quad (B.64)$$

Appendix C

\mathcal{H}_∞ AND MIXED $\mathcal{H}_2/\mathcal{H}_\infty$ –DESIGN PROBLEMS WITH MINIMUM FEEDBACK GAIN: THE STATE–FEEDBACK CASE

C.1 Introduction and Problem Formulation

Optimal \mathcal{H}_∞ –controllers may exhibit large feedback gains, resulting in large control efforts and increased noise sensitivity. Thus suboptimal controllers are normally preferred, since they do not usually exhibit these undesirable properties. However, the problem of high–gain occurrence may arise even in the suboptimal \mathcal{H}_∞ –design case and the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ –design problem when the \mathcal{H}_∞ –bound is too tight. Here the problem of designing a *minimum gain* static full state–feedback controller is considered for the same class of single–plant systems $\Sigma_{2/\infty,op,SF}$ as in section 6.2.1.

$$\Sigma_{2/\infty,op,SF} : \begin{cases} \dot{x}(t) &= Ax(t) + B_1w(t) + B_3u(t) \\ z_2(t) &= C_1x(t) + D_{13}u(t) \\ z_\infty(t) &= C_2x(t) + D_{23}u(t) \\ y(t) &= x(t) \end{cases} \quad (C.1)$$

with the same signal interpretations, signal dimensions and assumptions stated in section 6.2.1. Note that a non–zero direct feedthrough matrix from $w(t)$ to $z_\infty(t)$ can be incorporated into this framework as well. However, without loss of generality we assume this feedthrough matrix to be zero (see section 6.2.1).

For this type of systems a stabilizing static state–feedback gain $C_0 = D_c$ is sought such that the (nominal) Froebenius–norm of the feedback–gain matrix $\|C_0\|_F = \sqrt{\text{Trace}\{C_0C_0^T\}}$ is minimized and an \mathcal{H}_∞ –constraint on the closed–loop transfer function $T_\infty(C_0, s)$ from $w(s)$ to $z_\infty(s)$ is satisfied. With the cost function (4.4) in chapter 4 and the discussion in chapter 6 it is clear that the \mathcal{H}_∞ –bound can be replaced by a scalar convex cost function utilizing the controller parametrization $C_0 = WX^{-1}$ in [58]. However, the factorization $C_0 = WX^{-1}$ renders the criterion for the Froebenius–

norm $\|WX^{-1}\|_F$ of the feedback-gain matrix non-convex. By minimizing an upper bound for this cost, however, the problem can then be cast into a finite-dimensional optimization problem that is jointly convex in W and X . Finally it will be illustrated how to apply this theory to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -control problem with minimum gain considerations where additional \mathcal{H}_2 -performance specifications are considered for the transfer function $T_2(C_0, s)$ from $w(s)$ to $z_2(s)$.

Given a static state-feedback matrix C_0 , the closed-loop system $\Sigma_{2/\infty, cl, SF}$ is given by (6.23). For $\Sigma_{2/\infty, cl, SF}$ the design objective of a *minimum gain* \mathcal{H}_∞ -problem is then defined as follows:

- Design Problem P1:

Find a stabilizing state-feedback gain matrix C_0 such that $\|T_\infty(C_0)\|_\infty < \gamma$ and (an upper bound for) $\|C_0\|_F$ is minimized.

We also address the problem where $\|C_0\|_F$ is not actually minimized but bounded from above by a certain prespecified value b_C . Hence we can define an alternate criterion as follows:

- Design Problem P1':

Find a stabilizing state-feedback gain matrix C_0 such that $\|T_\infty(C_0)\|_\infty < \gamma$ and $\|C_0\|_F < b_C$.

The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -control problem with minimum gain can be put into the following form:

- Design Problem P2:

Find a stabilizing state-feedback gain matrix C_0 such that $\|T_\infty(C_0)\|_\infty < \gamma$ and an upper bound for the weighted sum of $\|T_2(C_0)\|_2$ and $\|C_0\|_F$ is minimized.

Similar to the objective in P1' we can also include design objectives where bounds are imposed on $\|T_2(C_0)\|_2$ and/or $\|C_0\|_F$. These cases will be outlined later. Note that all problems involving an \mathcal{H}_∞ -bound have a solution if and only if the associated pure \mathcal{H}_∞ -problem has a solution as shown in [140]. Design strategies that include either a bound on $\|C_0\|_F$ as in P1' or a bound on $\|T_2(C_0)\|_2$ such that $\|T_2(C_0)\|_2 < b_2$

may not have a solution even if the corresponding pure \mathcal{H}_∞ -bound problem has a solution. Note that with the above problem definitions the term “performance” can refer to either a pure \mathcal{H}_2 -criterion, the Froebenius norm of the state-feedback gain, or a weighted sum of both. In general all the above objectives will also reduce the control effort according to $\|u\|_2 \leq \|C_0\|_F \|x\|_2$.

With the controller factorization $C_0 = WX^{-1}$, $X = X^T > 0$ and the results in section 6.2.1 it is clear that the \mathcal{H}_∞ -bound can be replaced by an ARI-inequality and hence the scalar criterion

$$\lim_{t_{f1} \rightarrow \infty} \text{Trace}\{ e^{ARI_{C,SF}(W,X,\gamma)t_{f1}} \} = 0 \quad (\text{C.2})$$

where $ARI_{C,SF}(W, X, \gamma)$ is defined in (6.29). An upper bound for the \mathcal{H}_2 -norm of $T_2(C_0, s)$ is given by $J_{2,SF}(W, X)$ in (6.31). Furthermore, given a t_{f1} , both $J_{2,SF}(W, X)$ and $\text{Trace}\{ e^{ARI_{C,SF}(W,X,\gamma)t_{f1}} \}$ are jointly convex in W and X .

The cost associated with the Froebenius norm of the state-feedback gain matrix $C_0 = WX^{-1}$ on the other hand is not convex. Hence in order to arrive at an optimization problem that is jointly convex in the design parameters, one has to either find a different controller parametrization for the controller C_0 such that all the relevant cost functions are convex under this new factorization. This is in general a difficult problem. Alternatively one can maintain the factorization $C_0 = WX^{-1}$ for which $\text{Trace}\{ e^{ARI_{C,SF}(W,X,\gamma)t_{f1}} \}$ and $J_{2,SF}(W, X)$ are known to be convex and (similar to the upper bound for the \mathcal{H}_2 -cost) define an upper bound for the gain criterion $\|WX^{-1}\|_F$ that is also jointly convex in W and X . Furthermore, this upper bound should be as tight as possible to the true cost $\|WX^{-1}\|_F$ to avoid unnecessary conservatism. In the next section such a convex upper bound for $\|WX^{-1}\|_F$ is defined that allows the formulation of the above objectives in P1, P1' and P2 as convex optimization problems.

C.2 Convex Upper Bounds for $\|WX^{-1}\|_F$

Theorem C.2.1

Consider the Froebenius norm of the state-feedback gain matrix $\|C_0\|_F = \|WX^{-1}\|_F$ with $X = X^T > 0$ and a positive scalar τ , then the two scalar functions

$$J_{B1}(W, X, \tau) = \frac{1}{2}\tau^2\bar{\lambda}(X^{-1}) + \frac{1}{2}\text{Trace}(W^TW) \quad (\text{C.3})$$

$$J_{B2}(W, X, \tau) = \frac{1}{2}\text{Trace}(\tau^2X^{-1}) + \frac{1}{2}\text{Trace}(W^TW) \quad (\text{C.4})$$

with $\tau^2X \geq I$ represent upper bounds for $\|WX^{-1}\|_F$ such that

$$\|WX^{-1}\|_F \leq J_{B1}(W, X, \tau) \leq J_{B2}(W, X, \tau). \quad (\text{C.5})$$

Furthermore, $J_{B1}(W, X, \tau)$ and $J_{B2}(W, X, \tau)$ are jointly convex on W , $X = X^T > 0$ and $\tau > 0$. Moreover, an equivalent form for $\tau^2X \geq I$ is given by the matrix inequality constraint $\frac{1}{\tau^2}I - X \leq 0$ which is jointly convex on $X = X^T > 0$ and $\tau > 0$.

Proof: The following chain of inequalities proves that $J_{B1}(W, X, \tau)$ and $J_{B2}(W, X, \tau)$ represent upper bounds for $\|WX^{-1}\|_F$.

$$\|WX^{-1}\|_F = \sqrt{\text{Trace}(WX^{-1}X^{-1}W^T)} \quad (\text{C.6})$$

$$\leq \sqrt{\text{Trace}(\tau^2X^{-1}W^TW)} \quad (\text{C.7})$$

$$\text{s. t. } \tau^2X \geq I$$

$$\leq \sqrt{\tau^2\bar{\lambda}(X^{-1})\text{Trace}(W^TW)} \quad (\text{C.8})$$

$$\leq \frac{1}{2}\tau^2\bar{\lambda}(X^{-1}) + \frac{1}{2}\text{Trace}(W^TW) \quad (\text{C.9})$$

$$\leq \frac{1}{2}\text{Trace}(\tau^2X^{-1}) + \frac{1}{2}\text{Trace}(W^TW) \quad (\text{C.10})$$

provided that $\frac{1}{\tau^2}I - X \leq 0$. Obviously equation (C.9) is equivalent to $J_{B1}(W, X, \tau)$ and equation (C.10) represents $J_{B2}(W, X, \tau)$. Equation (C.7) follows from (C.6) by the scaling of $\|WX^{-1}\|_F$ with $\tau^2X \geq I$. (C.8) follows from (C.7) using lemma A.1.2 in appendix A. (C.9) follows from (C.8) using the arithmetic-geometric mean inequality with $\alpha = \frac{1}{2}$ and the facts that $\bar{\lambda}(\tau^2X^{-1}) \geq 0$ and $\text{Trace}(W^TW) \geq 0$ (see appendix A). $J_{B1}(W, X, \tau) \leq J_{B2}(W, X, \tau)$ finally follows from $\bar{\lambda}(Z) \leq \text{Trace}(Z)$ for any symmetric positive-definite matrix Z . Convexity of $\text{Trace}(W^TW)$ is shown in

appendix A (see lemma A.1.7) along with the other convexity proofs (see theorems A.1.2 and A.1.3). As the sum of convex-valued functions is convex, overall convexity follows. ■

Unfortunately no explicit expression for the gaps between the desired cost $\|C_0\|_F = \|WX^{-1}\|_F$ and the upper bounds $J_{B1}(W, X, \tau)$ and $J_{B2}(W, X, \tau)$ have been found. However, the additional constraint $\tau^2 X \geq I$ is not only necessary for $J_{B1}(W, X, \tau)$ and $J_{B2}(W, X, \tau)$ to be upper bounds for $\|WX^{-1}\|_F$, the additional optimization variable τ can be used to reduce the gap between the expressions (C.6) and (C.7) and hence the gaps between $\|WX^{-1}\|_F$ and the upper bounds. Note that $\frac{1}{\tau^2}I - X \leq 0$ is equivalent to $\tau^2 X \geq I$. The constraint $\frac{1}{\tau^2}I - X \leq 0$, however, is jointly convex on $X = X^T > 0$ and $\tau > 0$ while no proof for convexity of $\tau^2 X - I \geq 0$ has been found at this point.

Both bounds are continuous in W , X and τ . The function $J_{B1}(W, X, \tau)$ is obviously a tighter bound than $J_{B2}(W, X, \tau)$. However, $J_{B2}(W, X, \tau)$ is differentiable for all W , $X = X^T > 0$ and $\tau > 0$ while $J_{B1}(W, X, \tau)$ is not differentiable at points where $\bar{\lambda}(X^{-1}) = \lambda_i(X^{-1}) = \lambda_j(X^{-1})$, $i \neq j$. This property is important in the numerical solution of the minimization problem. Convexity of the cost functions $J_{B1}(W, X, \tau)$ and $J_{B2}(W, X, \tau)$ now allows us to redefine the design objectives P1 and P2 in terms of constrained convex optimization problems. Using the upper bounds derived above, the cost function representation for the \mathcal{H}_∞ -bound, the constraint $\frac{1}{\tau^2}I - X \leq 0$ and the upper bound for the \mathcal{H}_2 -cost, the design objectives can be reformulated as follows.

- P1: Minimum Gain Control with an \mathcal{H}_∞ -Bound:

$$\min_{W, X, \tau} J_{P1}(W, X, \tau) \quad (C.11)$$

$$J_{P1}(W, X, \tau) = J_{Bi}(W, X, \tau), \quad i = 1 \text{ or } i = 2$$

subject to

$$\lim_{t_{f1} \rightarrow \infty} \text{Trace} \{ e^{AR_{IC, SF}(W, X, \tau)t_{f1}} \} = 0 \quad (C.12)$$

$$\lim_{t_{f2} \rightarrow \infty} \text{Trace} \{ e^{(\frac{1}{\tau^2}I - X)t_{f2}} \} = 0 \quad (C.13)$$

$$\lim_{t_{f3} \rightarrow \infty} \text{Trace} \{ e^{-Xt_{f3}} \} = 0 \quad (C.14)$$

$$\tau > 0. \quad (C.15)$$

• P2: Minimum Gain Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -Control:

$$\min_{W, X, \tau} J_{P2}(W, X, \tau) \quad (\text{C.16})$$

$$J_{P2}(W, X, \tau) = \beta J_{Bi}(W, X, \tau) + (1 - \beta) J_{2,SF}(W, X),$$

$$i = 1 \text{ or } i = 2$$

subject to

$$\lim_{t_{f1} \rightarrow \infty} \text{Trace} \{ e^{ARI_{C,SF}(W,X,\tau)t_{f1}} \} = 0 \quad (\text{C.17})$$

$$\lim_{t_{f2} \rightarrow \infty} \text{Trace} \{ e^{(\frac{1}{\tau^2}I - X)t_{f2}} \} = 0 \quad (\text{C.18})$$

$$\lim_{t_{f3} \rightarrow \infty} \text{Trace} \{ e^{-Xt_{f3}} \} = 0 \quad (\text{C.19})$$

$$\tau > 0. \quad (\text{C.20})$$

where $\beta \in [0, 1]$ is a weighting factor. For $\beta = 0$ only the (nominal) \mathcal{H}_2 -performance measure is taken into consideration, with $\beta = 1$ the (nominal) minimum gain control problem is addressed.

Design objectives such as $\|C_0\|_F < b_C$ or $\|T_2(C_0)\|_2 < b_2$ can be incorporated in terms of $J_{Bi}(W, X, \tau)$, $i = 1$ or $i = 2$ and $J_{2,SF}(W, X)$ with the additional constraints

$$J_{Bi}(W, X, \tau) < b_C, \quad i = 1, 2 \quad (\text{C.21})$$

$$J_{2,SF}(W, X) < b_2. \quad (\text{C.22})$$

These constraints in turn can be replaced by the equivalent scalar representation

$$\lim_{t_{f4} \rightarrow \infty} \text{Trace} \{ e^{[J_{Bi}(W,X,\tau) - b_C]t_{f4}} \} = 0, \quad i = 1, 2 \quad (\text{C.23})$$

$$\lim_{t_{f5} \rightarrow \infty} \text{Trace} \{ e^{[J_{2,SF}(W,X) - b_2]t_{f5}} \} = 0, \quad (\text{C.24})$$

which can be appended to the above optimization problems. These objectives, however, are not investigated further in this work.

C.3 Numerical Treatment and Gradient Expressions

The function $J_{B1}(W, X, \tau)$ is generally not differentiable. Hence, if one chooses this function as the upper bound for the minimum gain problem, Ellipsoid or Cutting-Plane methods have to be applied to solve the problem at hand. For the numerical

example the upper bound $J_{B2}(W, X, \tau)$ has been chosen. This cost is differentiable as long as $X > 0$ and $\tau > 0$. Hence, gradient-based methods can be used to solve the optimization problem using this cost function. However, the constraints $X > 0$ and $\tau > 0$ are continuity constraints in this formulation and have to be satisfied throughout the optimization (see the discussion in chapter 5).

Algorithmically the problem can be solved in the same fashion as the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -problem in chapter 6. That is, it can be solved iteratively as a sequence of constrained optimization problems or, by introducing an overall cost function, as a sequence of unconstrained minimization problems. The reader is referred to the discussion in chapter 5 and chapter 6 regarding this procedure. Using the tools developed in appendix A, the following gradient expressions can be derived.

$$1. \quad \underline{J_{B2}(W, X, \tau) = \frac{1}{2}\text{Trace}(\tau^2 X^{-1}) + \frac{1}{2}\text{Trace}(W^T W) :}$$

$$\frac{\partial J_{B2}(W, X, \tau)}{\partial W} = W \quad (\text{C.25})$$

$$\frac{\partial J_{B2}(W, X, \tau)}{\partial X} = -\frac{1}{2}\tau^2 X^{-2} \quad (\text{C.26})$$

$$\frac{\partial J_{B2}(W, X, \tau)}{\partial \tau} = \text{Trace}(\tau X^{-1}) \quad (\text{C.27})$$

$$2. \quad \underline{\text{Trace} \{e^{(\frac{1}{\tau^2} I - X)t_{f2}}\} :}$$

$$\frac{\partial \text{Trace} \{e^{(\frac{1}{\tau^2} I - X)t_{f2}}\}}{\partial W} = 0 \quad (\text{C.28})$$

$$\frac{\partial \text{Trace} \{e^{(\frac{1}{\tau^2} I - X)t_{f2}}\}}{\partial X} = -t_{f2} e^{(\frac{1}{\tau^2} I - X)t_{f2}} \quad (\text{C.29})$$

$$\frac{\partial \text{Trace} \{e^{(\frac{1}{\tau^2} I - X)t_{f2}}\}}{\partial \tau} = -2 \frac{t_{f2}}{\tau^3} e^{(\frac{1}{\tau^2} I - X)t_{f2}} \quad (\text{C.30})$$

All other gradient expressions necessary to solve the above optimization problems (C.11) – (C.15) and (C.16) – (C.20) for $i = 2$ can be found in appendix B.

C.4 Example

To illustrate this approach, consider the 4th-order system used in [101] and in section 6.2.1. It represents the scaled subsystem of the lateral dynamics of a B-767 aircraft:

$$A = \begin{pmatrix} -0.0168 & 0.1121 & 0.0003 & -0.5608 \\ -0.0164 & -0.7771 & 0.9945 & 0.0015 \\ -0.0417 & -3.6595 & -0.9544 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} -0.0243 \\ -0.0634 \\ -3.6942 \\ 0 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}, \quad D_{13} = 1,$$

$$C_2 = \begin{pmatrix} 0.01 & 0 & 0.01 & 0 \end{pmatrix}, \quad D_{23} = 0.01.$$

The open-loop system is stable and the subsystem $T_\infty(C_0, s)$ has invariant zeros in the right-half plane. The minimally achievable $\|T_\infty(C_0)\|_\infty$ is approximately 0.007 and the minimally achievable \mathcal{H}_2 -norm $\|T_2(C_0)\|_2$ is 0.0078.

In Figure C.1 two curves are plotted. The design points on the upper curve ('o') represent the achieved controller gains $\|C_0\|_F = \|WX^{-1}\|_F$ by solving the convex optimization problem P1 defined in (C.11) – (C.15) with $J_{B2}(W, X, \tau)$ as performance cost for a given \mathcal{H}_∞ -bound γ_{spec} . The design points on the lower curve ('*') represent the resulting $\|C_0\|_F$ by solving the non-convex optimization problem P_{true} :

$$\min_{W, X} \|C_0\|_F \tag{C.31}$$

$$\text{subject to : } \|T_\infty(C_0)\|_\infty < \gamma_{spec} \tag{C.32}$$

in terms of the trace cost function associated with the general matrix inequality $ARI_C(C_0, X, \gamma_{spec}) < 0$ (see chapter 5). Hence no particular parametrization for C_0

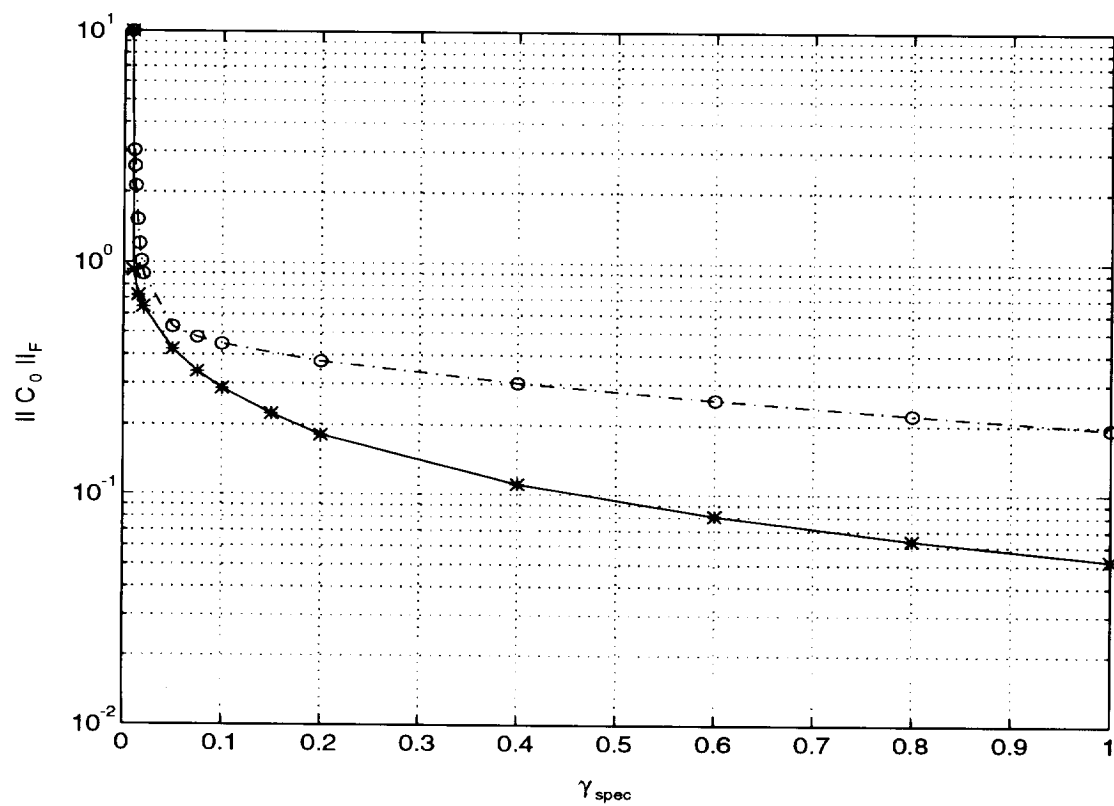


Figure C.1: \mathcal{H}_∞ -constrained minimum gain problem: Gain/robustness-tradeoff characteristics: $\|C_0\|_F$ versus specified \mathcal{H}_∞ -bound γ_{spec} ; Problem P1 with $J_{B2}(W, X, \tau)$, $C_0 = WX^{-1}$, $\beta = 1$: identified by 'o'; Problem P_{true}: identified by '*'.

is assumed for the design points on the lower curve in Figure C.1. Both curves show a typical behavior for mixed performance/robustness design objectives. For large γ_{spec} , $\|C_0\|_F$ is very small. If the overall problem becomes unconstrained in terms of the \mathcal{H}_∞ -constraint, that is, if γ_{spec} is chosen large enough, $\|C_0\|_F$ will converge to zero for both design curves (as the open-loop plant is stable for this example). For small γ_{spec} on the other hand a dramatic increase in the controller gain can be observed in both cases. The difference between the two curves illustrates the conservatism of the upper bound $J_{B2}(W, X, \tau)$ in comparison to the “true” (non-convex) optimization problem P_{true} , where no specific controller parametrization and no upper bound for $\|C_0\|_F$ was used. Note that the corresponding controller gains are plotted versus γ_{spec} and not the achieved \mathcal{H}_∞ -norm $\|T_\infty(C_0)\|_\infty$. For small γ_{spec} the achieved $\|T_\infty(C_0)\|_\infty$ is equal to the specified \mathcal{H}_∞ -bound. However, for large γ_{spec} this is no longer true. In these cases the achieved $\|T_\infty(C_0)\|_\infty$ were strictly smaller than γ_{spec} , confirming the conservatism of the \mathcal{H}_∞ -bound characterization $ARI_{C,SF}(C_0 = WX^{-1}, \gamma) < 0$ in terms of W and X as discussed in section 6.2.1

Figure C.2 shows the conservatism of the upper bound $J_{B2}(W, X, \tau)$ (upper curve, ‘x’) that was used as performance cost in problem P1 in comparison to the actually achieved norm of the controller gain $\|WX^{-1}\|_F$ (lower curve, ‘o’). Note that $J_{B2}(W, X, \tau)$ and $\|WX^{-1}\|_F$ are plotted on a logarithmic scale, showing that the convex upper bound $J_{B2}(W, X, \tau)$ for $\|WX^{-1}\|_F$ is rather tight for all design points.

Figures C.3 through C.5 show results related to solving problem P2, namely the minimum gain mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -control problem. In figure C.3 two design curves $\|T_2(C_0)\|_2$ versus γ_{spec} are shown that correspond to the design objective in P2 with $J_{B2}(W, X, \tau)$, $\beta = 0$ (pure \mathcal{H}_2 -problem, lower curve, ‘o’) and $\beta = 0.5$ (mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -control with minimum gain, upper curve, ‘*’) as performance costs respectively. Figure C.4 shows the corresponding controller gains for these design cases. The curves in Figures C.3 through C.5 again display the typical tradeoff characteristics between performance and stability robustness where performance in this case can be either the \mathcal{H}_2 -objective $\|T_2(C_0)\|_2$, the controller gain $\|WX^{-1}\|_F$, or a weighted sum of both. It is clear from these figures that for $\beta = 0$ the best \mathcal{H}_2 -norm characteristic (as a function of γ_{spec}) is achieved. For $\beta = 0.5$, when the controller gain is taken into consideration, the \mathcal{H}_2 -performance worsens which is reflected by larger \mathcal{H}_2 -norms $\|T_2(C_0)\|_2$ for $\beta = 0.5$ in comparison to those for $\beta = 0$. The resulting controller

gains on the other hand exhibit the reversed behavior (Figure C.4). This result is expected and it is further illustrated in Figure C.5 where plots are shown corresponding to the achieved controller gains for the design objective P2 with $J_{B2}(W, X, \tau)$, $\beta = 0$ (pure \mathcal{H}_2 -problem, upper curve, ‘o’), $\beta = 0.5$ (mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -control with minimum gain, center curve, ‘*’) and $\beta = 1$ (\mathcal{H}_∞ -constrained minimum gain control, lower curve, ‘+’) respectively. Figure C.5 illustrates that an increase in β implies a decrease in the controller gains. Note also that for $\beta = 1$ (lower curve) the controller gains converge to zero for large γ_{spec} . This is due to the fact the the open-loop system is stable. Hence if γ_{spec} is larger than the open-loop \mathcal{H}_∞ -norm $\|T_\infty(C_0)\|_\infty$, a controller $C_0 = WX^{-1}$ with $W = 0$ will “stabilize” the plant and satisfy the specified \mathcal{H}_∞ -bound implying $\|WX^{-1}\|_F = 0$. If \mathcal{H}_2 -objectives are incorporated ($\beta = 0$, $\beta = 0.5$), then the controller gains do not tend to zero for large γ_{spec} as in this case the additional performance cost corresponding to $\|T_2(C_0)\|_2$ would not be minimal.

The actual choice of the controller depends on the chosen performance specifications in terms of the controller gains and the \mathcal{H}_2 -performance cost as well as the necessary \mathcal{H}_∞ -robust stability requirements.

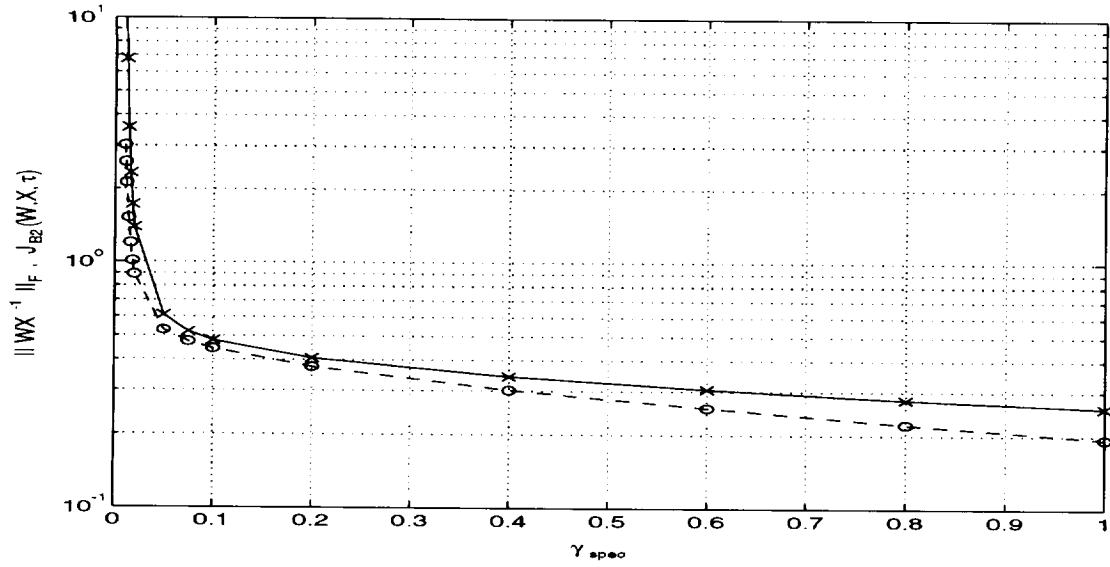


Figure C.2: \mathcal{H}_∞ -constrained minimum gain problem P1 with $J_{B2}(W, X, \tau)$, $C_0 = WX^{-1}$, $\beta = 1$: $\|WX^{-1}\|_F$ ('o') and the upper bound $J_{B2}(W, X, \tau)$ ('x') versus the specified \mathcal{H}_∞ -bound γ_{spec} .

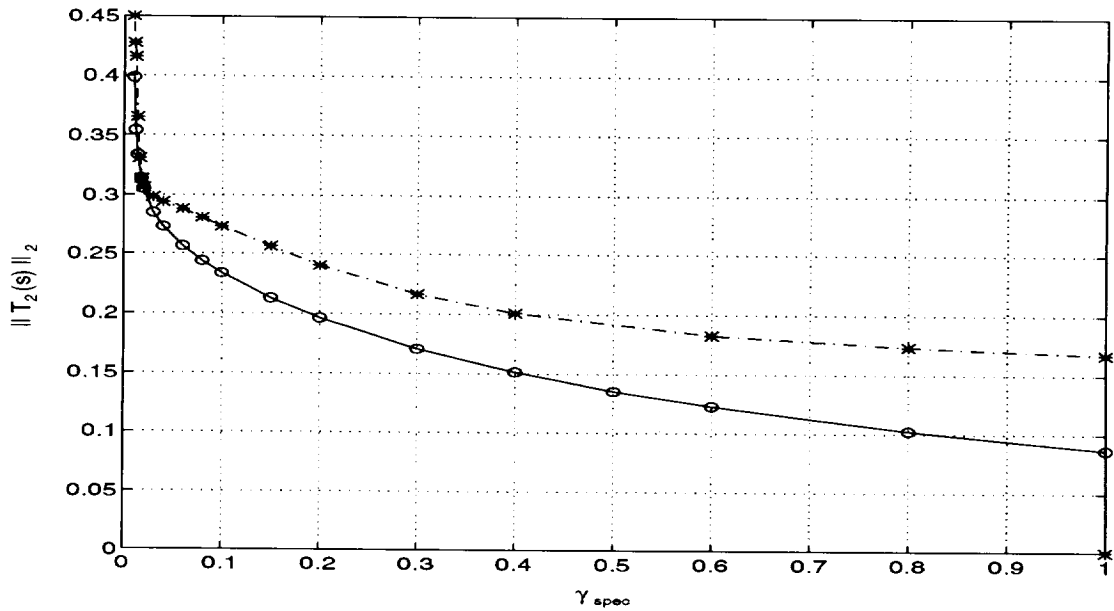


Figure C.3: Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -control with minimum gain: \mathcal{H}_2 /robustness tradeoff characteristics: \mathcal{H}_2 -performance versus specified \mathcal{H}_∞ -bound γ_{spec} ; Problem P2 with $J_{B2}(W, X, \tau)$, $\beta = 0$ ('o') and $\beta = 0.5$ (*).

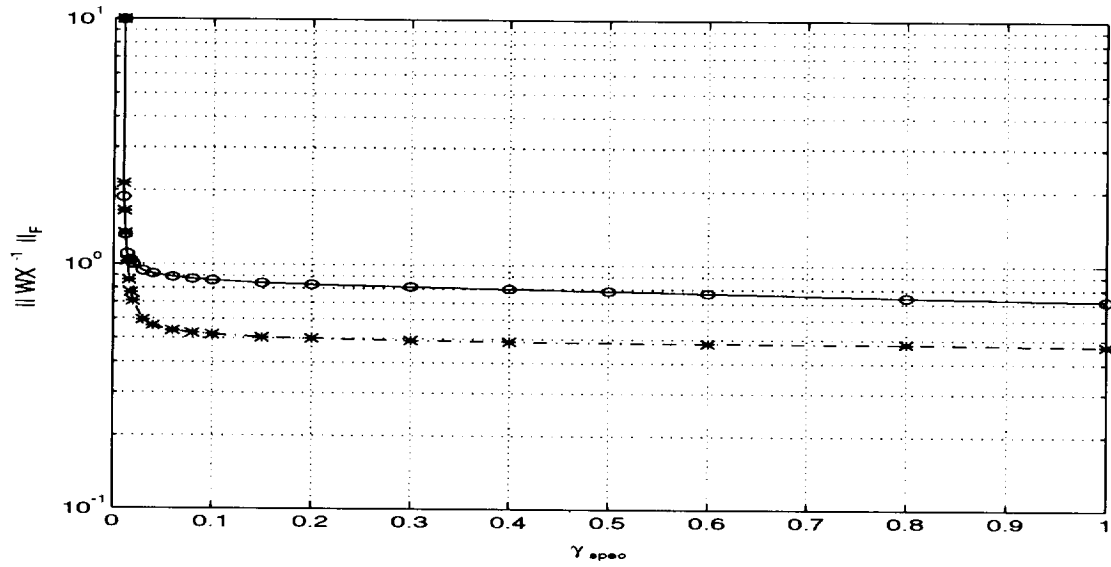


Figure C.4: $\|WX^{-1}\|_F$ versus specified \mathcal{H}_∞ -bound γ_{spec} ; mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -control with minimum gain; Problem P2 with $J_{B2}(W, X, \tau)$, $\beta = 0$ ('o') and $\beta = 0.5$ ('*').

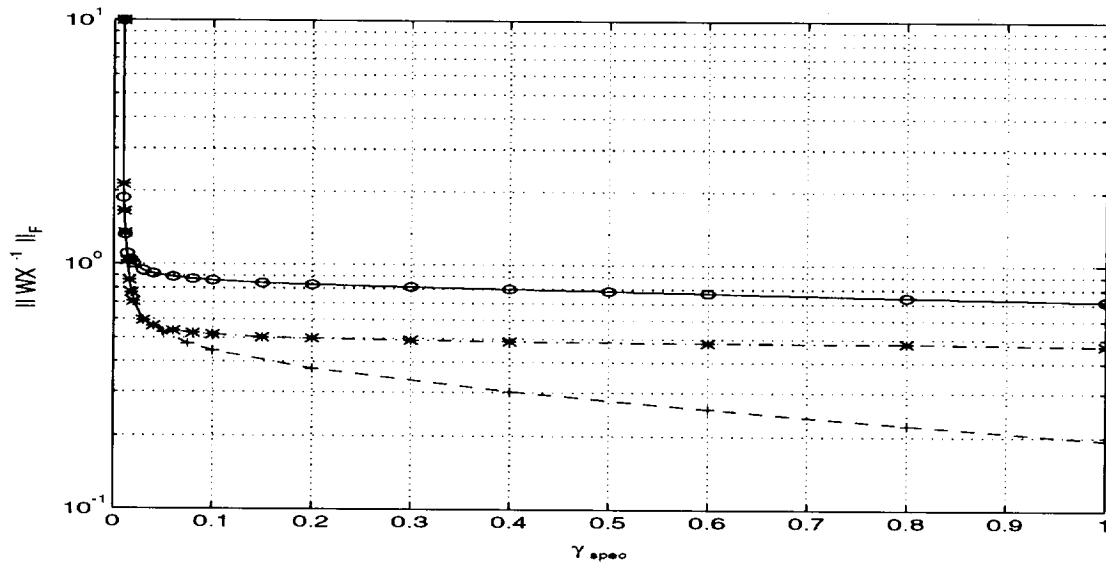


Figure C.5: $\|WX^{-1}\|_F$ versus specified \mathcal{H}_∞ -bound γ_{spec} ; mixed $\mathcal{H}_2/\mathcal{H}_\infty$ -control with minimum gain; Problem P2 with $J_{B2}(W, X, \tau)$, $\beta = 0$ ('o'), $\beta = 0.5$ ('*') and $\beta = 1$ ('+').

Appendix D

ROBUST STATE-FEEDBACK CONTROLLERS FOR SYSTEMS UNDER MIXED TIME/FREQUENCY-DOMAIN CONSTRAINTS

In this appendix the tools developed in chapters 4 and 5 and in appendix C are applied to design problems that involve time-domain constraints on the control action and the closed-loop system state as well as frequency-domain \mathcal{H}_∞ -constraints. The problem is formulated in the discrete-time domain and hence extends further the results presented previously. In order to emphasize the discrete-time domain in this appendix, all signals are identified accordingly, i.e., $x(k)$ or $u(k)$. Associated transfer functions in this domain are identified by their dependence on ‘ z ’, the variable of the z -transform. For related norm-definitions of discrete-time signals the reader is referred to [62] or [72].

In the following discussion a system $G := (A, B, C, D)$ denotes a linear, shift-invariant, discrete-time system

$$G : \begin{cases} x(k+1) &= Ax(k) + Bw(k) \\ y(k) &= Cx(k) + Dw(k) \end{cases} \quad (\text{D.1})$$

Such a system $G := (A, B, C, D)$ is asymptotically stable if all the eigenvalues λ_i of A satisfy $|\lambda_i| < 1$. The \mathcal{H}_∞ -norm for a discrete-time system $G := (A, B, C, D)$ is defined as

$$\|G(z)\|_\infty = \sup_{\theta \in [0, 2\pi]} \bar{\sigma}[G(z = e^{j\theta})] \quad (\text{D.2})$$

where $G(z)$ is the transfer function from $w(z)$ to $y(z)$ associated with the system $G := (A, B, C, D)$.

D.1 Introduction and Problem Formulation

A large number of control problems require designing a controller capable of achieving acceptable performance in the presence of system uncertainty and to given design specifications usually described in both the time and frequency-domains. However, despite its practical importance, this problem still remains to a large extent unsolved, even in the simpler case where the system under consideration is linear. During the last decade a large research effort has led to procedures for designing robust controllers capable of achieving desirable properties under various classes of model uncertainties. The \mathcal{H}_∞ -framework, combined with μ -analysis ([22], in order to exploit the structure of the uncertainty) has been successfully applied to a number of hard practical control problems (see for instance [110]). However, in spite of this success, it is clear that plain \mathcal{H}_∞ -control can only address a subset of the common performance requirements since, being a frequency-domain method, it cannot address time-domain specifications. Some approaches that incorporate time-domain constraints into the \mathcal{H}_∞ -formalism have been recently developed ([109], [95], [122]). However, these approaches require solving large, non-differentiable optimization problems and typically result in a very large controller order, necessitating some type of model reduction ([122]).

A different approach to robust control has been pursued in [131], [74], where robustness and disturbance rejection are approached using the l_1 -optimal control theory introduced by Vidyasagar ([131]) and developed by Pearson and coworkers ([74]). These methods are attractive since they allow for an explicit solution to the robust performance problem. However, they cannot accommodate some common classes of frequency-domain specifications (such as \mathcal{H}_2 or \mathcal{H}_∞ -bounds).

Finally, a third approach to controlling time-domain constrained systems exploits the concept of positively invariant sets ([10], [123], [121], [130]). Although this approach leads to simple design algorithms and has recently been extended to encompass some robustness considerations, it cannot handle frequency-domain specifications.

In the following an approach is presented that satisfies certain time-domain constraints by converting these constraints to problems where set-induced operator norms are minimized (or bounded from above). The frequency-domain constraints

considered here are robust stability criteria expressed in term of \mathcal{H}_∞ -constraints. Note that \mathcal{H}_2 -performance measures can be incorporated in the same way as in appendix C. However, this will not be considered further here. Consider the following linear, shift-invariant, discrete-time system

$$\Sigma_{D,\infty,op,SF} : \begin{cases} x(k+1) &= Ax(k) + B_1w(k) + B_3u(k) \\ z_\infty(k) &= C_2x(k) + D_{23}u(k) \\ y(k) &= x(k) \end{cases} \quad (D.3)$$

where (A, B_3) is controllable, D_{23} has full column rank, $x(k) \in R^{n_x}$, $w(k) \in R^{n_w}$, $z_\infty(k) \in R^{n_{z_\infty}}$ and $u(k) \in R^{n_u}$. System uncertainties are assumed to be lumped into the system $\Delta(z)$ with $w(z) = \Delta(z)z_\infty(z)$. The controller under consideration is a static full state-feedback controller $C_0 = D_c$ realizing the control law $u(k) = C_0y(k)$. Given a state-feedback matrix C_0 , the closed-loop system can be expressed as follows:

$$\Sigma_{D,\infty,cl,SF} : \begin{cases} x_{cl}(k+1) &= A_{cl}x_{cl}(k) + B_1w(k) \\ &= (A + B_3C_0)x_{cl}(k) \\ z_\infty(k) &= C_{cl,\infty}x_{cl}(k) \\ &= (C_2 + D_{23}C_0)x_{cl}(k) \end{cases} \quad (D.4)$$

where $A_{cl} = A + B_3C_0$. Let $T_\infty(C_0, z)$ denote the closed-loop transfer function from $w(z)$ to $z_\infty(z)$. In face of equation (D.4) we can state the design objectives of the design problem considered here as follows.

$P_{\mathcal{H}_\infty, T}$: \mathcal{H}_∞ -Robust Control Problem with Time-Domain Constraints:

Given the system $\Sigma_{D,\infty,op,SF}$ and two convex, compact, balanced sets ([62]) containing the origin in their interior, $\mathcal{W} \subset R^{n_x}$ and $\mathcal{U} \subset R^{n_u}$, find a stabilizing static state-feedback gain matrix C_0 such that:

$$\|T_\infty(C_0, z)\|_\infty \leq \gamma \quad (D.5)$$

and, for the nominal system $\Sigma_{D,\infty,cl,SF}$ with $w(k) = 0 \ \forall k$,

$$x_{cl}(k) \in \mathcal{W}, \ \forall k, \quad (D.6)$$

$$u(k) \in \mathcal{U}, \ \forall k. \quad (D.7)$$

Time-domain constraints such as (D.6) or (D.7) have important implications in plants where the closed-loop system states $x_{cl}(k)$ and the control $u(k)$ are required to remain within certain safety limits. Typical examples for such plants are boiler systems or power plants. Note that the specific type of time-domain constraints depends on the choice of the spaces \mathcal{W} and \mathcal{U} respectively. Constraint (D.5) on the other hand enforces robust stability with respect to the uncertainties $\Delta(z)$ in terms of an \mathcal{H}_∞ -constraint on $T_\infty(C_0, z)$. To establish the connection between the problem of minimizing an induced operator norm and the time-domain constraints (D.6) and (D.7) a result concerning constrained control problems is recalled ([121]).

Definition D.1.1 ([62])

The Minkowsky functional $p(x_{cl})$ of a balanced convex set \mathcal{W} containing the origin in its interior is defined in terms of a real scalar parameter $r > 0$ by

$$p(x_{cl}) = \inf_{r>0} \left\{ r: \frac{x_{cl}(k)}{r} \in \mathcal{W} \right\}. \quad (\text{D.8})$$

A well-known result in functional analysis (see for instance [62]) establishes that $p(x_{cl}(k))$ defines a seminorm in R^{n_x} . Furthermore, when \mathcal{W} is compact, this seminorm becomes a norm. This result is exploited in the following lemma.

Lemma D.1.1 ([121])

Consider the system:

$$x_{cl}(k+1) = A_{cl}x_{cl}(k) \quad (\text{D.9})$$

and let $\|\cdot\|_{\mathcal{W}}$ denote the operator norm induced in $R^{n_x \times n_x}$ by \mathcal{W}

(i.e. $\|A_{cl}\|_{\mathcal{W}} \triangleq \sup_{\|x_{cl}(k)\|_{\mathcal{W}}=1} \|A_{cl}x_{cl}(k)\|_{\mathcal{W}}$). Then, given \mathcal{W} and an initial condition $x_{cl}(k=0) \in \mathcal{W}$, the trajectory $x_{cl}(k) \in \mathcal{W}$ for all k if and only if $\|A_{cl}\|_{\mathcal{W}} \leq 1$.

This lemma shows that the time-domain constraints (D.6) and (D.7) can be expressed equivalently in terms of bounds on the induced operator norms $\|A_{cl}\|_{\mathcal{W}}$ and $\|C_0\|_{\mathcal{W}, \mathcal{U}}$ respectively where $\|C_0\|_{\mathcal{W}, \mathcal{U}} \triangleq \sup_{\|x_{cl}(k)\|_{\mathcal{W}} \leq 1} \|C_0x_{cl}\|_{\mathcal{U}}$.

$$\|A_{cl}\|_{\mathcal{W}} \leq 1 \quad (\text{D.10})$$

$$\|C_0\|_{\mathcal{W}, \mathcal{U}} \leq 1. \quad (\text{D.11})$$

Moreover, it can be shown ([123], [124]) that minimizing $\|A_{cl}\|_{\mathcal{W}}$ maximizes robustness against parametric model uncertainty and minimizes the effects of the disturbance $w(k)$. In the next section it is shown how to utilize lemma D.1.1 and the controller parametrization $C_0 = WX^{-1}$ to convert problem $P_{\mathcal{H}_{\infty},T}$ to a convex suboptimal optimization problem.

D.2 Reformulation of the Design Problem as a Convex Optimization Problem

Using the controller factorization $C_0 = WX^{-1}$ as in section 6.2.1, corresponding discrete-time ARI (LMI)-criteria for the \mathcal{H}_{∞} -bound have been established for the full state-feedback case and are as follows.

Lemma D.2.1 ([55])

Consider the asymptotically stable system $\Sigma_{D,\infty,cl,SF}$. Assume that (A_{cl}, B_3) is controllable and $(C_{cl,\infty}, A_{cl})$ is observable. Then the following statements are equivalent:

1. $\|T_{\infty}(C_0, z)\|_{\infty} < \gamma$

2. *ARI: There exists a symmetric positive definite matrix Y such that*

$$ARI_{D,SF}(C_0, Y, \gamma) < 0 \quad (D.12)$$

$$\begin{aligned} ARI_{D,SF}(C_0, Y, \gamma) := & A_{cl}YA_{cl}^T - Y + B_1B_1^T \\ & + A_{cl}YC_{cl,\infty}^T[M(C_0, Y, \gamma)]^{-1}C_{cl,\infty}YA_{cl}^T, \end{aligned} \quad (D.13)$$

$$M(C_0, Y, \gamma) = \gamma^2 I - C_{cl,\infty}YC_{cl,\infty}^T. \quad (D.14)$$

3. *There exists a symmetric positive definite matrix X such that*

$$LMI_{D,SF}(C_0, X, \gamma) < 0 \quad (D.15)$$

$$\begin{aligned} LMI_{D,SF}(C_0, X, \gamma) := & \begin{pmatrix} A_{cl} \\ C_{cl,\infty} \end{pmatrix} X \begin{pmatrix} A_{cl}^T & C_{cl,\infty}^T \end{pmatrix} \\ & + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \begin{pmatrix} B_1^T & 0 \end{pmatrix} - \begin{pmatrix} X & 0 \\ 0 & \gamma^2 I \end{pmatrix}. \end{aligned} \quad (D.16)$$

Lemma D.2.2 ([55])

Consider the same system as in lemma D.2.1 and let $C_0 = WX^{-1}$ with $K \in R^{n_u \times n_x}$ and $X = X^T > 0 \in R^{n_x \times n_x}$, then the matrix mapping $LMI_{D,SF}(C_0, X, \gamma) = LMI_{D,SF}(C_0 = WX^{-1}, X, \gamma) = LMI_{D,SF}(W, X, \gamma)$ with

$$\begin{aligned} LMI_{D,SF}(W, X, \gamma) &:= \begin{pmatrix} A + B_3WX^{-1} \\ C_2 + D_{23}WX^{-1} \end{pmatrix} X \begin{pmatrix} A + B_3WX^{-1} \\ C_2 + D_{23}WX^{-1} \end{pmatrix}^T \\ &\quad + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} (B_1^T \quad 0) - \begin{pmatrix} X & 0 \\ 0 & \gamma^2 I \end{pmatrix} \end{aligned} \quad (D.17)$$

is jointly convex on W and X . Furthermore, there exists a static state-feedback $C_0 = WX^{-1}$ such that $\|T_\infty(C_0, z)\|_\infty < \gamma$ if and only if there are W and $X = X^T > 0$ such that $LMI_{D,SF}(W, X, \gamma) < 0$.

Hence, even in the discrete-time domain, the controller factorization $C_0 = WX^{-1}$ provides a means to represent the \mathcal{H}_∞ -constraint in terms of the convex matrix inequality $LMI_{D,SF}(W, X, \gamma) < 0$. By utilizing the cost function in chapter 4 it is then known that the matrix constraint $LMI_{D,SF}(W, X, \gamma) < 0$ and hence the \mathcal{H}_∞ -constraint $\|T_\infty(C_0, z)\|_\infty < \gamma$ can be substituted by a convex scalar constraint

$$\lim_{t_{f1} \rightarrow \infty} \text{Trace} \{ e^{LMI_{D,SF}(W, X, \gamma)t_{f1}} \} = 0. \quad (D.18)$$

The equivalent time-domain constraints (D.10) and (D.11) in terms of the induced operator norms on the other hand depend on the chosen spaces \mathcal{W} and \mathcal{U} and are not in general convex if the controller factorization $C_0 = WX^{-1}$ is selected. However, since all finite-dimensional matrix norms are equivalent ([49]), it follows that there exist constants c_1 and c_2 , depending only on the geometry of the sets \mathcal{W} and \mathcal{U} , such that

$$\|\cdot\|_{\mathcal{W}} \leq c_1 \|\cdot\|_F \quad (D.19)$$

$$\|\cdot\|_{\mathcal{W}\mathcal{U}} \leq c_2 \|\cdot\|_F. \quad (D.20)$$

Hence, suboptimal time-domain constraints for the inequalities (D.10) and (D.11) can be defined in terms of the Froebenius norm of the matrices A_{cl} and C_0 . To this type of cost function the results in appendix C are applicable to define convex

upper bounds for $\|A_{cl}\|_F$ and $\|C_0\|_F$ using the factorization $C_0 = WX^{-1}$ for which the \mathcal{H}_∞ -constraint can be represented by a convex constraint. Note that only the differentiable upper bound developed in appendix C will be used here. Utilizing the triangular inequality, the submultiplicativity property of the Froebenius norm, the controller factorization $C_0 = WX^{-1}$ and the results in appendix C, it is easily verified that

$$\|A_{cl}\|_F = \|A + B_3WX^{-1}\|_F \quad (\text{D.21})$$

$$\leq \|A\|_F + \|B_3WX^{-1}\|_F \quad (\text{D.22})$$

$$\leq \|A\|_F + \frac{1}{2}\text{Trace}(\tau^2 X^{-1}) + \frac{1}{2}\text{Trace}(W^T B_3^T B_3 W) \quad (\text{D.23})$$

where $\tau > 0$ and $\frac{1}{\tau^2}I - X \leq 0$ are assumed to hold. Joint convexity of $\frac{1}{2}\text{Trace}(\tau^2 X^{-1})$ and the constraint $\frac{1}{\tau^2}I - X \leq 0$ in $\tau > 0$ and $X = X^T > 0$ have been shown in appendix C. Convexity of the term $\text{Trace}(W^T B_3^T B_3 W)$ follows immediately from lemma A.1.7 in appendix A as $B_3^T B_3 \geq 0$. With $\|A\|_F$ being a constant, overall joint convexity of the right hand side of (D.23) on W , $X = X^T > 0$ and τ follows immediately. Hence with the results in appendix C and with the factorization $C_0 = WX^{-1}$ we have arrived at the following upper bounds for the induced norm inequalities (D.10) and (D.11).

$$\frac{1}{c_1}\|A_{cl}\|_W \leq \|A_{cl}\|_F \quad (\text{D.24})$$

$$\leq \|A\|_F + \frac{1}{2}\text{Trace}(\tau^2 X^{-1}) + \frac{1}{2}\text{Trace}(W^T B_3^T B_3 W) \quad (\text{D.25})$$

and

$$\frac{1}{c_2}\|C_0\|_{W,U} = \frac{1}{c_2}\|WX^{-1}\|_{W,U} \quad (\text{D.26})$$

$$\leq \|B_3WX^{-1}\|_F \quad (\text{D.27})$$

$$\leq \frac{1}{2}\text{Trace}(\tau^2 X^{-1}) + \frac{1}{2}\text{Trace}(W^T W). \quad (\text{D.28})$$

With these upper bounds, the cost function defined in chapter 4 and the abbreviations

$$J_P(W, X, \tau) = \frac{1}{2}\text{Trace}(\tau^2 X^{-1}) + \frac{1}{2}\text{Trace}(W^T B_3^T B_3 W) \quad (\text{D.29})$$

$$J_P(B_3 W, X, \tau) = \frac{1}{2}\text{Trace}(\tau^2 X^{-1}) + \frac{1}{2}\text{Trace}(W^T W) \quad (\text{D.30})$$

we are now in the position to reformulate problem $P_{\mathcal{H}_\infty, T}$ as a convex suboptimal optimization problem.

$P_{\mathcal{H}_\infty, T}^{conv}$: Convex Suboptimal \mathcal{H}_∞ -Robust Control with Time-Domain Constraints:

Given the system $\Sigma_{D, \infty, op, SF}$, find W and $X = X^T > 0$ such that:

$$1.) \quad \lim_{t_{f1} \rightarrow \infty} \text{Trace} \{ e^{LM I_{D, SF}(W, X, \gamma) t_{f1}} \} = 0 \quad (\text{D.31})$$

$$2.) \quad \lim_{t_{f2} \rightarrow \infty} \text{Trace} \{ e^{-X t_{f2}} \} = 0 \quad (\text{D.32})$$

$$3.) \quad \lim_{t_{f3} \rightarrow \infty} \text{Trace} \{ e^{[J_P(B_3 W, X, \tau) - \frac{1}{c_1}] t_{f3}} \} = 0 \quad (\text{D.33})$$

$$4.) \quad \lim_{t_{f4} \rightarrow \infty} \text{Trace} \{ e^{[J_F(W, X, \tau) - \frac{b_u}{c_2}] t_{f4}} \} = 0 \quad (\text{D.34})$$

$$5.) \quad \tau > 0. \quad (\text{D.35})$$

where b_u is the maximum control effort allowed. Note that the performance functional $J_P(B_3 W, X, \tau)$ does not include the term $\|A\|_F$ which is part of the upper bound (D.23). This is justified by the fact that $\|A\|_F$ does not depend on any of the optimization variables and thus is constant. Once such a controller has been found, the actual (nominal) bounds on the desired time-domain constraints (D.6) and (D.7) follow from the chosen spaces \mathcal{W} and \mathcal{U} and the corresponding constants c_1 and c_2 in (D.19) and (D.20).

D.3 Gradient Expressions

The suboptimal reformulation of the original problem $P_{\mathcal{H}_\infty, T}$ in terms of $P_{\mathcal{H}_\infty, T}^{conv}$ in (D.31) – (D.35) is generally a multi-objective problem. Numerically it can be solved using the same tools as in appendix C or in chapter 5. That is, it can be solved either as a sequence of constrained optimization problems where one selects one of the costs in (D.31) – (D.35) as performance costs and treats all the other criteria as constraints. Alternatively one can form an overall cost function (see chapter 5) and solve the problem as a sequence of unconstrained minimization problems for increasing values of t_{fi} ($i = 1, 2, 3, 4$). The problem is in general differentiable, where $\tau > 0$ and $X > 0$ have to be enforced as continuity constraints throughout the numerical optimization (see discussion in chapter 5). Note that the constraint (D.31) enforces the block-structured matrix inequality $LM I_{D, SF}(W, X, \gamma) < 0$. Hence, to derive gradients for

the cost function associated with the constraint (D.31) the results for this type of matrix inequality constraints in chapter 4 have to be utilized. Following these results and the framework in appendix A and appendix B, gradients for the cost functions associated with the problem $P_{\mathcal{H}_{\infty},T}^{conv}$ are as follows.

1. $Trace \{ e^{LMI_{D,SF}(W,X,\gamma)t_{f1}} \}$:

Let

$$E_{D,SF} = e^{LMI_{D,SF}(W,X,\gamma)t_{f1}} = \begin{pmatrix} E_{D,SF,11} & E_{D,SF,12} \\ E_{D,SF,21} & E_{D,SF,22} \end{pmatrix}, \quad (D.36)$$

where the partitioning of $E_{D,SF}$ corresponds to the block-structure of $LMI_{D,SF}(W, X, \gamma)$, then

$$\frac{\partial Trace \{ e^{LMI_{D,SF}(W,X,\gamma)t_{f1}} \}}{\partial W} = 2 t_{f1} \begin{pmatrix} B_3 \\ D_{23} \end{pmatrix}^T E_{D,SF} \begin{pmatrix} A_{cl} \\ C_{cl,\infty} \end{pmatrix} \quad (D.37)$$

$$\begin{aligned} \frac{\partial Trace \{ e^{LMI_{D,SF}(W,X,\gamma)t_{f1}} \}}{\partial X} &= t_{f1} \{ E_{D,SF,11} + \begin{pmatrix} A \\ C_2 \end{pmatrix}^T E_{D,SF} \begin{pmatrix} A \\ C_2 \end{pmatrix} \\ &\quad - X^{-1} W \begin{pmatrix} B_3 \\ D_{23} \end{pmatrix}^T E_{D,SF} \begin{pmatrix} B_3 \\ D_{23} \end{pmatrix} W X^{-1} \} \end{aligned} \quad (D.38)$$

$$\frac{\partial Trace \{ e^{LMI_{D,SF}(W,X,\gamma)t_{f1}} \}}{\partial \tau} = 0. \quad (D.39)$$

2. $Trace \{ e^{[J_P(B_3 W, X, \tau) - \frac{1}{c_1}]t_{f3}} \}$:

$$\frac{\partial Trace \{ e^{[J_P(B_3 W, X, \tau) - \frac{1}{c_1}]t_{f3}} \}}{\partial W} = t_{f3} B_3^T B_3 W e^{[J_P(B_3 W, X, \tau) - \frac{1}{c_1}]t_{f3}} \quad (D.40)$$

$$\frac{\partial Trace \{ e^{[J_P(B_3 W, X, \tau) - \frac{1}{c_1}]t_{f3}} \}}{\partial X} = -t_{f3} \tau^2 X^{-2} e^{[J_P(B_3 W, X, \tau) - \frac{1}{c_1}]t_{f3}} \quad (D.41)$$

$$\frac{\partial Trace \{ e^{[J_P(B_3 W, X, \tau) - \frac{1}{c_1}]t_{f3}} \}}{\partial \tau} = t_{f3} \tau Trace(X^{-1}) e^{[J_P(B_3 W, X, \tau) - \frac{1}{c_1}]t_{f3}}. \quad (D.42)$$

3. Trace $\{e^{[J_P(W,X,\tau)-\frac{b_u}{c_2}]t_{f4}}\}$:

$$\frac{\partial \text{Trace} \{e^{[J_P(W,X,\tau)-\frac{b_u}{c_2}]t_{f4}}\}}{\partial W} = t_{f4} W e^{[J_P(W,X,\tau)-\frac{b_u}{c_2}]t_{f4}} \quad (\text{D.43})$$

$$\frac{\partial \text{Trace} \{e^{[J_P(W,X,\tau)-\frac{b_u}{c_2}]t_{f4}}\}}{\partial X} = -t_{f4} \tau^2 X^{-2} e^{[J_P(W,X,\tau)-\frac{b_u}{c_2}]t_{f4}} \quad (\text{D.44})$$

$$\frac{\partial \text{Trace} \{e^{[J_P(W,X,\tau)-\frac{b_u}{c_2}]t_{f4}}\}}{\partial \tau} = 2 t_{f4} \left(\frac{t_{f4}}{\tau^3}\right) e^{[J_P(W,X,\tau)-\frac{b_u}{c_2}]t_{f4}}. \quad (\text{D.45})$$

The gradients for all the other cost functions associated with the constraints (D.31) – (D.35) have been derived in appendix B and appendix C.

D.4 Example

The approach is illustrated on a discretized 4th-order system representing the lateral dynamics of a B-767 aircraft. The (continuous-time) model is the same as in section 6.2.1 and appendix C. For this plant the state-space matrices are given as follows:

$$A = \begin{pmatrix} 0.9966 & 0.0227 & -0.0084 & -0.1120 \\ -0.0037 & 0.7952 & 0.1633 & 0.0005 \\ -0.0063 & -0.6008 & 0.7661 & 0.0003 \\ -0.0007 & -0.0645 & 0.1779 & 1.0000 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0.1885 \\ -0.0003 \\ -0.0007 \\ 0.2000 \end{pmatrix}, \quad B_3 = \begin{pmatrix} -0.0029 \\ -0.0762 \\ -0.6529 \\ -0.0683 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 0.0100 & 0 & 0.0100 & 0 \end{pmatrix}, \quad D_{23} = 0.0100.$$

The open-loop system is stable, the open-loop \mathcal{H}_∞ -norm $\|T_\infty(C_0, z)\|_\infty$ is 7.4826, the minimally achievable norm $\|T_\infty(C_0, z)\|_\infty$ is approximately 0.007 and the Froeb-nius norm of the open-loop system-matrix A is $\|A\|_F = 1.9102$. For this plant the following optimization has been solved for various \mathcal{H}_∞ -bounds γ_{spec} .

$$\min_{W, X, \tau} J_P(B_3W, X, \tau) + \|A\|_F \quad (\text{D.46})$$

subject to

$$\lim_{t_{f1} \rightarrow \infty} \text{Trace} \{ e^{LMI_{D, SF}(W, X, \gamma)t_{f1}} \} = 0 \quad (\text{D.47})$$

$$\lim_{t_{f2} \rightarrow \infty} \text{Trace} \{ e^{-Xt_{f2}} \} = 0 \quad (\text{D.48})$$

$$\tau > 0. \quad (\text{D.49})$$

Thus, for this example, we actually minimize the upper bound $J_P(B_3W, X, \tau) + \|A\|_F$ rather than bounding it from above. Hence the mixed performance/robustness problem solved here uses the upper bound $J_P(B_3W, X, \tau) + \|A\|_F$ as the performance cost. Robust stability is incorporated by an \mathcal{H}_∞ -bound in terms of the constraints (D.47) and (D.48). This implies an optimization on possible (nominal) time-domain constraints on $x_{cl}(k)$. Possible time-domain constraints on $u(k)$ are not explicitly incorporated in this example. Figure D.1 displays the results for the optimization problem (D.46) – (D.49) with the given plant. The upper curve ('*') plots the upper bound cost $J_P(B_3W, X, \tau) + \|A\|_F$ versus the specified \mathcal{H}_∞ -bound γ_{spec} . The center curve ('o') and the lower curve ('+') reflect the resulting Froebenius norms of the closed-loop matrix A_{cl} ($\|A_{cl}\|_F$) and the controller gain WX^{-1} ($\|WX^{-1}\|_F$) respectively.

All curves display the usual performance/robustness tradeoff characteristics. Interesting is the fact that $\|A_{cl}\|_F$ decreases monotonically up to $\gamma_{spec} = 0.1$. For $\gamma_{spec} > 0.1$ $\|A_{cl}\|_F$ increases and converges to the open-loop norm $\|A\|_F$. This fact is due to the stability of the open-loop system. Note, that for $\gamma_{spec} \geq 7.4826$, $W = 0$ will satisfy the required \mathcal{H}_∞ -bound and “stabilize” the system and $\|A_{cl}\|_F = \|A\|_F = 1.9102$ in this case. The achieved controller gain $\|WX^{-1}\|_F$ on the other hand shows the typical performance/stability robustness tradeoff and exhibits the behavior already encountered in the equivalent continuous-time example (see section 6.2.1 and appendix C). Depending on the type of time-domain constraints on $x_{cl}(k)$, the chosen spaces \mathcal{W} and \mathcal{U} and hence the constants c_1 and c_2 one would now select one of the design points to derive the actually achieved bounds on the desired time-domain constraints.

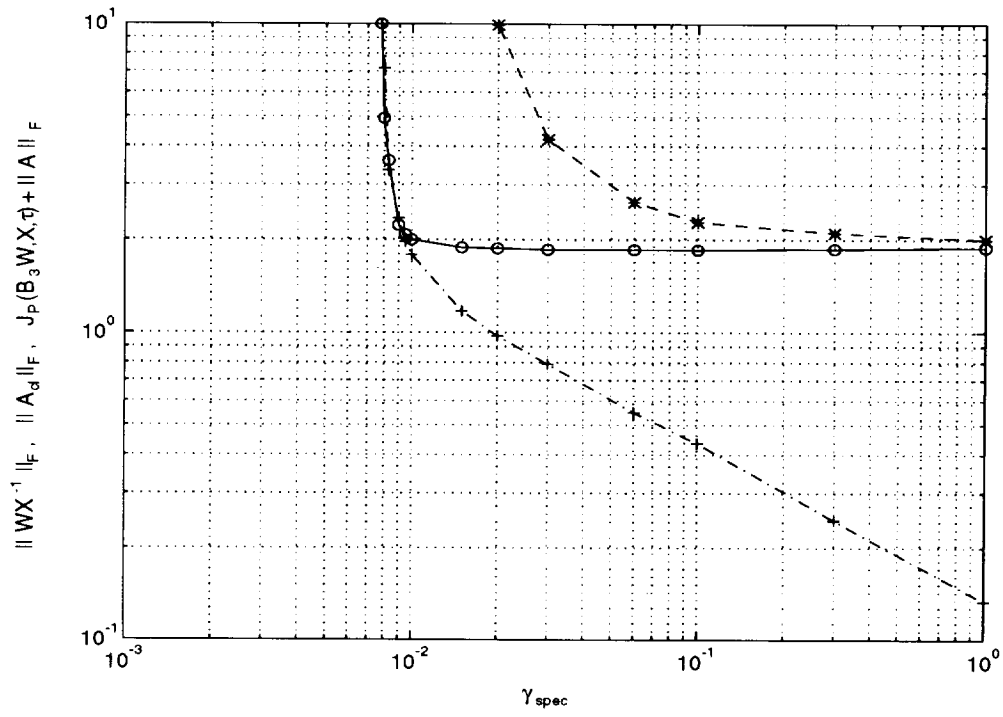


Figure D.1: \mathcal{H}_∞ -constrained control with time-domain constraints: $J_P(B_3W, X, \tau) + \|A\|_F$: ('*'); $\|A_{cl}\|_F$: ('o'); $\|WX^{-1}\|_F$: ('+').

Appendix E

MULTI-PLANT \mathcal{H}_∞ -DESIGN SOFTWARE

The underlying approach for the \mathcal{H}_∞ -design software is equivalent to that presented in chapter 5 except for the notational convention. That is, n_p open-loop plants of the following form are considered here.

$$\Sigma_{\infty,op}^i : \begin{cases} \dot{\bar{x}}^i(t) &= A^i \bar{x}^i(t) + B_1^i u^i(t) + B_2^i w_\infty^i(t) \\ y^i(t) &= C_1^i \bar{x}^i(t) + D_{11}^i u^i(t) + D_{12}^i w_\infty^i(t) \\ z_\infty^i(t) &= C_2^i \bar{x}^i(t) + D_{21}^i u^i(t) + D_{22}^i w_\infty^i(t) \end{cases} \quad (\text{E.1})$$

for $i = 1, 2, \dots, n_p$. This notational convention is most widely used in literature (see e.g. [115]). For a given plant condition, model uncertainties are assumed to be lumped into a stable, norm-bounded $\Delta^i(s)$ -block,

$$\|\Delta^i(s)\|_\infty \leq \frac{1}{\gamma^i} \quad (\text{E.2})$$

with the usual feedback connection

$$w_\infty^i(s) = \Delta^i(s) z_\infty^i(s). \quad (\text{E.3})$$

The state vectors $x^i(t)$, the control inputs $u^i(t)$ and the disturbance input/criterion signals $w_\infty^i(t)$ and $z_\infty^i(t)$ are assumed to have the same dimensions as in chapter 5. That is, $x^i(t) \in R^{n_{x^i}}$, $w_\infty^i(t) \in R^{n_{w_\infty^i}}$, $z_\infty^i(t) \in R^{n_{z_\infty^i}}$, $u^i(t) \in R^{n_{u^i}}$ and $y^i \in R^{n_{y^i}}$ for $i = 1, 2, \dots, n_p$. All the involved matrices are assumed to have compatible dimensions. The controller $C(s)$ has the same form as in (3.4) with a corresponding parametric representation

$$C_0 = \begin{pmatrix} D_c & C_c \\ B_c & A_c \end{pmatrix}. \quad (\text{E.4})$$

With these definitions and the notation in this appendix, the system assumptions corresponding to the \mathcal{H}_∞ -design problem in chapter 3 are as follows.

Assumptions:

A1: (A^i, B_1^i) are stabilizable pairs for all $i = 1, 2, \dots, n_p$,

A2: (A^i, C_1^i) are detectable pairs for all $i = 1, 2, \dots, n_p$,

A3: $\dim(u^i) = n_{u^i} = n_u$ and $\dim(y^i) = n_{y^i} = n_y$ for all $i = 1, 2, \dots, n_p$,

A4: $D_{11}^1 = D_{11}^2 = \dots = D_{11}^{n_p} = D_{11}$.

These assumptions have been discussed in chapter 3. Given a controller C_o with the state-space realization (3.4), the corresponding closed-loop plants are given by

$$\Sigma_{\infty, cl}^i(C_o) : \begin{cases} \dot{x}^i(t) &= A_{cl, \infty}^i x^i(t) + B_{cl, \infty}^i w_\infty^i(t) \\ z_\infty^i(t) &= C_{cl, \infty}^i x^i(t) + D_{cl, \infty}^i w_\infty^i(t) \end{cases} \quad (\text{E.5})$$

for $i = 1, 2, \dots, n_p$, where (with the notation in this appendix) the closed-loop matrices are as follows,

$$\begin{aligned} A_{cl, \infty}^i &= \bar{A}^i + \bar{B}_1^i C_o \bar{C}_1^i \\ B_{cl, \infty}^i &= \bar{B}_2^i + \bar{B}_1^i C_o \bar{D}_{12}^i \\ C_{cl, \infty}^i &= \bar{C}_2^i + \bar{D}_{21}^i C_o \bar{C}_1^i \\ D_{cl, \infty}^i &= \bar{D}_{22}^i + \bar{D}_{21}^i C_o \bar{D}_{12}^i \end{aligned} \quad (\text{E.6})$$

where

$$\begin{aligned} \bar{A}^i &= \begin{pmatrix} A^i & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{B}_1^i = \begin{pmatrix} B_1^i & 0 \\ 0 & I \end{pmatrix}, \quad \bar{B}_2^i = \begin{pmatrix} B_2^i \\ 0 \end{pmatrix}, \\ \bar{C}_1^i &= \begin{pmatrix} C_1^i & 0 \\ 0 & I \end{pmatrix}, \quad \bar{C}_2^i = \begin{pmatrix} C_2^i & 0 \end{pmatrix}, \\ \bar{D}_{12}^i &= \begin{pmatrix} D_{12}^i \\ 0 \end{pmatrix}, \quad \bar{D}_{21}^i = \begin{pmatrix} D_{21}^i & 0 \end{pmatrix}, \quad \bar{D}_{22}^i = D_{22}^i. \end{aligned}$$

With the n_p closed-loop matrices $A_{cl, \infty}^i$, $B_{cl, \infty}^i$, $C_{cl, \infty}^i$ and $D_{cl, \infty}^i$ the n_p \mathcal{H}_∞ -constraints

$\|T_\infty^i(C_0, s)\|_\infty < \gamma^i$ are then equivalent to the system of matrix inequalities (3.30):

$$\begin{aligned} 1.) \quad & ARI_{C,OF}^i(C_0, X^i, \gamma^i) < 0 \\ 2.) \quad & D_{cl,\infty}^{iT} D_{cl,\infty}^i - (\gamma^i)^2 I < 0 \\ 3.) \quad & -X^i < 0 \\ 4.) \quad & X^i = X^{iT} \end{aligned} \tag{E.7}$$

where the corresponding matrix expressions in (E.7) have been defined in chapter 3. An important observation on a lower bound for all X^i has been made in theorem 2.2.1 of chapter 2, namely

$$X^i \geq L_o^i \tag{E.8}$$

where L_o^i is the observability grammian that satisfies

$$A_{cl,\infty}^{iT} L_o^i + L_o^i A_{cl,\infty}^i + C_{cl,\infty}^{iT} C_{cl,\infty}^i = 0, \tag{E.9}$$

assuming $A_{cl,\infty}^i$ is stable. Hence, solutions X^i satisfying (E.7) can be rewritten as $X^i = L_o^i + \tilde{X}^i + \tilde{X}^{iT} - \text{diag}(\tilde{X}^i)$ for a set of n_p upper triangular matrices \tilde{X}^i . This knowledge is explicitly incorporated into the design software. Rather than having a set of n_p optimization variables X^i , we optimize over n_p upper triangular matrices \tilde{X}^i (see (5.13)) from which X^i is formed by $X^i = L_o^i + \tilde{X}^i + \tilde{X}^{iT} - \text{diag}(\tilde{X}^i)$. The condition $-X^i < 0$ can then be replaced by the condition that $-(\tilde{X}^i + \tilde{X}^{iT} - \text{diag}(\tilde{X}^i)) < 0$ as L_o^i is positive semidefinite. Symmetry of X^i is explicitly taken into account in this formulation. The enforcement of the required stability constraints on $A_{cl,\infty}^i$ is described later.

To avoid local minima in this general purpose design package an additional constraint of

$$A_{aux}^i + B_{aux}^i X^i \text{ to be asymptotically stable} \tag{E.10}$$

with

$$A_{aux}^i = A_{cl}^i + B_{cl,\infty}^i (R^i)^{-1} (D_{cl,\infty}^i)^T C_{cl,\infty}^i \tag{E.11}$$

$$B_{aux}^i = B_{cl,\infty}^i (R^i)^{-1} (B_{cl,\infty}^i)^T \tag{E.12}$$

is enforced as well (see discussion in chapter 5). The set \mathcal{G} contains the specified \mathcal{H}_∞ -bounds γ^i and is defined in chapter 5 and for further reference the following set is introduced.

$$\tilde{\mathcal{X}} = \{ \tilde{X}^i : \tilde{X}^i \text{ upper triangular, } i = 1, 2, \dots, n_p \}. \tag{E.13}$$

E.1 Design Cost Function

With the above sets and abbreviations and the cost function defined in the body of this report the multi-plant \mathcal{H}_∞ -design problem stated in chapter 5 can now be restated as follows.

“Find an internally stabilizing controller C_0^* and a set $\tilde{\mathcal{X}}^*$ of n_p matrices \tilde{X}^{i*} that solve the following minimization problem.

$$J(C_0^*, \tilde{\mathcal{X}}^*) = \lim_{\substack{t_{fk}^i \rightarrow \infty \\ (k=1,2,3,4, i=1,\dots,n_p)}} \min_{C_0, \tilde{\mathcal{X}}} \sum_{i=1}^{n_p} J_{des}^i(C_0, \tilde{X}^i, t_{f1}^i, t_{f2}^i, t_{f3}^i, t_{f4}^i) \quad (\text{E.14})$$

$$J_{des}^i(C_0, \tilde{X}^i, t_{f1}^i, t_{f2}^i, t_{f3}^i, t_{f4}^i) = [f_{1,des}^i(C_0, \tilde{X}^i, t_{f1}^i) + f_{2,des}^i(C_0, t_{f2}^i) + f_{3,des}^i(\tilde{X}^i, t_{f3}^i) + f_{4,des}^i(\tilde{X}^i, t_{f4}^i)] \quad (\text{E.15})$$

$$f_{1,des}^i(C_0, \tilde{X}^i, t_{f1}^i) = \text{Trace} \{ e^{ARI^i(C_0, \tilde{X}^i)t_{f1}^i} \} \quad (\text{E.16})$$

$$f_{2,des}^i(C_0, t_{f2}^i) = \text{Trace} \{ e^{-[(\gamma^i)^2 I - D_{cl,\infty}^i D_{cl,\infty}^i] t_{f2}^i} \} \quad (\text{E.17})$$

$$f_{3,des}^i(\tilde{X}^i, t_{f3}^i) = \text{Trace} \{ e^{-[\tilde{X}^i + \tilde{X}^{iT} - \text{diag}(\tilde{X}^i)] t_{f3}^i} \} \quad (\text{E.18})$$

$$f_{4,des}^i(\tilde{X}^i, t_{f4}^i) = \sum_{k=1}^{n_{xi} + n_c} [(f_{4,des}^i)_k(\tilde{X}^i, t_{f4}^i)] \quad (\text{E.19})$$

$$(f_{4,des}^i)_k(\tilde{X}^i, t_{f4}^i) = \begin{cases} [e^{(\lambda_{re})_k^i t_{f4}^i} - 1]^2 & \text{if } (\lambda_{re})_k^i \geq 0 \\ 0 & \text{if } (\lambda_{re})_k^i < 0 \end{cases} \quad (\text{E.20})$$

where

$$ARI^i(C_0, \tilde{X}^i) = (X^i B_{cl,\infty}^i + C_{cl,\infty}^{iT} D_{cl,\infty}^i)(R^i)^{-1}(X^i B_{cl,\infty}^i + C_{cl,\infty}^{iT} D_{cl,\infty}^i)^T + A_{cl,\infty}^{iT} X^i + X^i A_{cl,\infty}^i + C_{cl,\infty}^{iT} C_{cl,\infty}^i$$

$$X^i = L_o^i + [\tilde{X}^i + \tilde{X}^{iT} - \text{diag}(\tilde{X}^i)],$$

\tilde{X}^i is upper triangular,

$A_{cl,\infty}^i$ is stable,

$$A_{cl,\infty}^{iT} L_o^i + L_o^i A_{cl,\infty}^i + C_{cl,\infty}^{iT} C_{cl,\infty}^i = 0,$$

$$[A_{aux}^i + B_{aux}^i X^i][(\lambda_{re})_k^i + j(\lambda_{im})_k^i] = [(\lambda_{re})_k^i + j(\lambda_{im})_k^i][(\lambda_{re})_k^i + j(\lambda_{im})_k^i].$$

If a solution can be found for C_0^* and \tilde{X}^* that yield a finite value of $J(C_0^*, \tilde{X}^*)$, then it is guaranteed that all the specified \mathcal{H}_∞ -constraints are satisfied. The overall number n_{var} of possible optimization variables in the minimization problem is

$$n_{var} = n_{con} + \sum_{i=1}^{n_p} \frac{1}{2} (n_{x^i} + n_c)(n_{x^i} + n_c + 1) \quad (\text{E.21})$$

where n_{con} is the number of selected optimization variables in C_0 .

In our approach the variables t_{fk}^i ($k = 1, 2, 3, 4$, $i = 1, \dots, n_p$) act as scaling variables in the overall cost function $\sum_{i=1}^{n_p} J_{des}^i(C_0, \tilde{X}^i, t_{f1}^i, t_{f2}^i, t_{f3}^i, t_{f4}^i)$. Starting with small values of t_{fk}^i we proceed to optimize the objective function $\sum_{i=1}^{n_p} J^i(C_0, \tilde{X}^i, t_{f1}^i, t_{f2}^i, t_{f3}^i, t_{f4}^i)$ until a reasonable convergence has been reached for these values of t_{fk}^i . The values t_{fk}^i are subsequently increased and the optimization is repeated. This process is continued until all the constraints have been successfully satisfied. In this case, a controller that satisfies all the \mathcal{H}_∞ -bounds has been found. The convergence often fails when the objective function cannot be improved further after the specified number of iterations has been exhausted. In this case the chosen \mathcal{H}_∞ -bounds may be either too small, or the desired controller structure is too restrictive.

E.2 Gradient Expressions

The gradient expressions for all relevant cost functions can be derived using the formalism in the appendices A and B. With the new characterization $X^i = L_o^i + [\tilde{X}^i + \tilde{X}^{iT} - \text{diag}(\tilde{X}^i)]$ the according gradient expressions are quite complex but are included at this point for the sake of completeness for the case $Cost_sel = 1$ (Trace-type cost function). Gradients for the cases $Cost_sel = 2$ and $Cost_sel = 3$ are easily found from the results for $Cost_sel = 1$ (see `Hi_des_grad.m`). Using the following definitions

$$E_1^i = e^{ARI^i(C_0, \tilde{X}^i)t_{f1}^i}, \quad (\text{E.22})$$

$$E_2^i = e^{-[(\gamma^i)^2 I - D_{cl,\infty}^{iT} D_{cl,\infty}^i]t_{f2}^i}, \quad (\text{E.23})$$

$$E_3^i = e^{-[\tilde{X}^i + \tilde{X}^{iT} - \text{diag}(\tilde{X}^i)]t_{f3}^i}, \quad (\text{E.24})$$

$$L_1^i = A_{cl,\infty}^i + B_{cl,\infty}^i (R^i)^{-1} D_{cl,\infty}^{iT} C_{cl,\infty}^i, \quad (\text{E.25})$$

$$L_2^i = B_{cl,\infty}^i (R^i)^{-1} B_{cl,\infty}^{iT}, \quad (\text{E.26})$$

$$L_3^i = C_{cl,\infty}^{iT} [D_{cl,\infty}^i (R^i)^{-1} D_{cl,\infty}^{iT} + I] C_{cl,\infty}^{iT}, \quad (\text{E.27})$$

$$L_4^i = X^i B_{cl,\infty}^i + C_{cl,\infty}^{iT} D_{cl,\infty}^i, \quad (\text{E.28})$$

$$L_5^i = (L_1^i + L_2^i X) E_1^i, \quad (\text{E.29})$$

$$L_6^i = [\bar{C}_1^i + \bar{D}_{12}^i (R^i)^{-1} L_4^{iT}] E_1^i, \quad (\text{E.30})$$

$$L_7^i = X^i \bar{B}_1^i + [C_{cl,\infty}^{iT} + L_4^i (R^i)^{-1} D_{cl,\infty}^{iT}] \bar{D}_{21}^i, \quad (\text{E.31})$$

$$L_8^i = E_1^i (A_{cl,\infty}^i + L_4^i (R^i)^{-1} B_{cl,\infty}^{iT}), \quad (\text{E.32})$$

$$L_9^i = L_8^i + L_8^{iT}, \quad (\text{E.33})$$

$$L_{10}^i : A_{cl,\infty}^i L_{10}^i + L_{10}^i A_{cl,\infty}^{iT} + L_9^i = 0, \quad (\text{E.34})$$

$$L_{11}^i : A_{cl,\infty}^{iT} L_{11}^i + L_{11}^i A_{cl,\infty}^i + C_{cl,\infty}^{iT} C_{cl,\infty}^i = 0, \quad (\text{E.35})$$

$$L_{12}^i = \bar{C}_1^i L_{10}^i (L_{11}^i \bar{B}_1^i + \bar{D}_{21}^i), \quad (\text{E.36})$$

$$L_{13}^i = \bar{D}_{21}^{iT} D_{cl,\infty}^i E_3^i \bar{D}_{12}^{iT} \quad (\text{E.37})$$

$$L_{14}^i = \frac{(q_2)_k^i}{(q_1^2)_k^i + (q_2^2)_k^i} [(u_{re})_k^i (v_{re}^T)_k^i + (u_{im})_k^i (v_{im}^T)_k^i] \\ + \frac{(q_1)_k^i}{(q_1^2)_k^i + (q_2^2)_k^i} [(u_{re})_k^i (v_{im}^T)_k^i - (u_{im})_k^i (v_{re}^T)_k^i] \quad (\text{E.38})$$

$$[L_1^i + L_2^i X^i] [(u_{re})_k^i + j(u_{im})_k^i] = [(\lambda_{re})_k^i + j(\lambda_{im})_k^i] [(u_{re})_k^i + j(u_{im})_k^i] \\ [L_1^i + L_2^i X^i]^T [(v_{re})_k^i + j(v_{im})_k^i] = [(\lambda_{re})_k^i - j(\lambda_{im})_k^i] [(v_{re})_k^i + j(v_{im})_k^i] \\ [(u_{re})_k^i - j(u_{im})_k^i]^T [(u_{re})_k^i + j(u_{im})_k^i] = 1 \\ [(v_{re})_k^i - j(v_{im})_k^i]^T [(v_{re})_k^i + j(v_{im})_k^i] = 1 \\ (q_1)_k^i = (v_{im}^T)_k^i (u_{re})_k^i - (v_{re}^T)_k^i (u_{im})_k^i \\ (q_2)_k^i = (v_{im}^T)_k^i (u_{im})_k^i + (v_{re}^T)_k^i (u_{re})_k^i$$

$$L_{15}^i = \bar{C}_1^i L_{15}^i + \bar{D}_{12}^i (R^i)^{-1} [D_{cl,\infty}^{iT} C_{cl,\infty}^i L_{15}^i + B_{cl,\infty}^i (L_{15}^{iT} X^i + X^i (L_{15}^i))] \quad (\text{E.39})$$

$$L_{16}^i = \bar{B}_1^i + B_{cl,\infty}^i (R^i)^{-1} D_{cl,\infty}^{iT} \bar{D}_{21}^i \quad (\text{E.40})$$

$$L_{17}^i = \bar{D}_{12}^i (R^i)^{-1} B_{cl,\infty}^{iT} L_{15}^i C_{cl,\infty}^{iT} [I + D_{cl,\infty}^i (R^i)^{-1} D_{cl,\infty}^{iT}] \bar{D}_{21}^i \quad (\text{E.41})$$

$$L_{18}^i = L_{15}^i L_2^i + L_2^i L_{15}^i \quad (\text{E.42})$$

$$L_{19}^i = L_{11}^i \quad (\text{E.43})$$

$$L_{20}^i : A_{cl,\infty}^i L_{20}^i + L_{20}^i A_{cl,\infty}^{iT} + L_{18}^i = 0, \quad (\text{E.44})$$

$$L_{21}^i = \bar{C}_1^i L_{20}^i [L_{19}^i \bar{B}_1^i + C_{cl,\infty}^{iT} \bar{D}_{21}^i] \quad (\text{E.45})$$

$$L_{22}^i = L_{15}^i L_{16}^i + L_{17}^i + L_{21}^i \quad (\text{E.46})$$

the gradient expressions can be written as follows,

$$\frac{\partial f_{1,des}^i(C_0, \tilde{X}^i, t_{f1}^i)}{\partial \tilde{X}^i} = t_{f1}^i (L_5^i + L_5^{iT}) \quad (\text{E.47})$$

$$\frac{\partial f_{1,des}^i(C_0, \tilde{X}^i, t_{f1}^i)}{\partial C_0} = 2t_{f1}^i (L_6^i + L_7^i + L_{12}^i)^T \quad (\text{E.48})$$

$$\frac{\partial f_{2,des}^i(C_0, t_{f2}^i)}{\partial C_0} = 2t_{f2}^i L_{13}^i \quad (\text{E.49})$$

$$\frac{\partial f_{3,des}^i(\tilde{X}^i, t_{f3}^i)}{\partial \tilde{X}^i} = -t_{f3}^i E_3^i \quad (\text{E.50})$$

$$\frac{\partial (f_{4,des}^i)_k(\tilde{X}^i, t_{f4}^i)}{\partial \tilde{X}^i} = \begin{cases} 2t_{f4}^i [e^{(\lambda_{re})_k^i t_{f4}^i} - 1] e^{(\lambda_{re})_k^i t_{f4}^i} L_{18}^i & \text{if } (\lambda_{re})_k^i \geq 0 \\ 0 & \text{if } (\lambda_{re})_k^i < 0 \end{cases} \quad (\text{E.51})$$

$$\frac{\partial (f_{4,des}^i)_k(\tilde{X}^i, t_{f4}^i)}{\partial C_0} = \begin{cases} 2t_{f4}^i [e^{(\lambda_{re})_k^i t_{f4}^i} - 1] e^{(\lambda_{re})_k^i t_{f4}^i} L_{22}^i & \text{if } (\lambda_{re})_k^i \geq 0 \\ 0 & \text{if } (\lambda_{re})_k^i < 0 \end{cases} \quad (\text{E.52})$$

All the other partial derivatives are zero. Gradients of the overall cost function are given by the summation over all the plant conditions of the individual gradients at each plant condition using equations (E.47)-(E.52). Note that the cost function $f_{4,des}^i(\tilde{X}^i, t_{f4}^i)$ is differentiable only if the according eigenvalues are simple. This formulation has worked well in a practical implementation. There may, however, be situations where the algorithm fails to converge due to the presence of parameter combination such that the cost function $f_{4,des}^i(\tilde{X}^i, t_{f4}^i)$ is not differentiable. To circumvent this problem, an additional set of parameters needs to be introduced (see discussion in chapter 5). Such additional variables, however, would increase the computational burden considerably and hence this route is not chosen in this design package.

E.3 Program Structure

The above formulation requires that all $A_{cl,\infty}^i$ are stable matrices. Hence before starting the optimization on the above overall cost function a controller has to be found that internally stabilizes all n_p plant conditions simultaneously. For this reason the

overall algorithm is divided into two phases as follows.

Phase 1:

Find a controller $C(s)$ that stabilizes all plant conditions simultaneously such that $\bar{\sigma}(D_{cl,\infty}^i) < \gamma^i$ for all plant conditions. This phase utilizes a cost function similar to that in (E.19) and (E.20) defined on the closed-loop matrices $A_{cl,\infty}^i$ (see `Hi_des.m` and `Hi_des_func.m`).

Phase 2:

Optimize on $\sum_{i=1}^{n_p} J_{des}^i(C_0, \tilde{\mathcal{X}}^i, t_{f1}^i, t_{f2}^i, t_{f3}^i, t_{f4}^i)$ for increasing values $t_{f1}^i, t_{f2}^i, t_{f3}^i$ and t_{f4}^i until all \mathcal{H}_∞ -constraints are satisfied.

E.4 Program Description

E.4.1 Software Requirements and Global Variables

The included MATLAB-files are written for the MATLAB-version 4.1 and require the following additional toolboxes:

1. Control Systems Toolbox
2. Robust Control Toolbox
3. Optimization Toolbox

MATLAB 4.1 offers the capability of limiting the scope of the global variables. Namely, if a variable is not defined as a global variable inside a function, then it is considered a local variable there.

In MATLAB 3.xx a global variable is global everywhere. Hence, if the user is interested in running `Hi_des.m` on MATLAB 3.xx, the removal of the global statements in all functions is a possibility. The variables that are required globally have then to be defined as global variables before calling `Hi_des.m`. These global variables have the following names:

- *Global_var* *n_p* *n_u* *n_y* *n_c* *N_var* *N_varX* *N_varCo* *N_varX_ind*
- *System_1* *System_2* *System_3* ...

- *Dimensions Dimensions_1 Dimensions_2 Dimensions_3 ...*
- *Co_fix Covar_fix C_optim Co_var_Indexset Xvar_fix X_optim*
- *D_optim Stab_bound Stab_only Cost_sel*

These variables will be erased from the global workspace and should not be used in the external program calling `Hi_des.m`.

E.4.2 MATLAB Functions

The software includes the following MATLAB-files:

- `Hi_des_inpu.m`
- `Hi_des.m`
- `Hi_des_func.m`, `Hi_des_grad.m`
- `Hi_des_upda.m`, `Hi_des_tfup.m`
- `Hi_des_opti.m`, `Hi_des_qupr.m`, `Hinorm_comp.m`

A copy of this software can be obtained directly from the authors. For this reason a listing of these files is not included in this work. All of these programs are MATLAB-functions, that is, except for the global variables, they do not share common variables. All functions have an extensive header containing information on the purpose of the input and output variables, and the required global variables. This information can be retrieved by typing “`help Hi_des.m`”, “`help Hi_des_inpu.m`”, ... and so on. In the following we will describe each file separately.

1. `Hi_des_inpu.m`

This function requires the user to enter the plant data for each plant condition. Except for the data storage, this function does not have any input or output arguments. Before this routine is called, the user should have the following data ready for input:

- A filename under which the plant data will be stored. The data will be stored in the chosen filename plus the extension *_hi.mat*. For example, if the chosen filename is “Test”, then the data will be stored in a MATLAB data file named “Test_hi.mat”.
- n_p is the number of plant conditions.
- n_y and n_u are the dimensions of the measurement vector and control vector respectively. Note that these two dimensions must be the same for all plant conditions.
- n_{x^i} , $n_{w_\infty^i}$ and $n_{z_\infty^i}$ are respectively the dimension of the state-space plant model, the dimension of the disturbance vector w_∞^i , and the dimension of the criterion vector z_∞^i for the i^{th} plant condition.
- The system matrices for each plant condition are saved in the following form:

$$System_i = \begin{pmatrix} A^i & B_1^i & B_2^i \\ C_1^i & D_{11}^i & D_{12}^i \\ C_2^i & D_{21}^i & D_{22}^i \end{pmatrix}$$

Note that the matrices D_{11}^i are required to be identical for all plant conditions. It is suggested that the user stores these matrices in a *xxx.mat* file which can be retrieved during an input session. The variable names for these matrices should however not conflict with the global variables as defined above.

2. **Hi_des.m**

This function contains the main program. This is the function to be called for the actual design of a controller. Its input and output arguments as well its internal organization are discussed below.

3. **Hi_des_func.m**

This function computes the function value $\sum_{i=1}^{n_p} J_{des}^i(C_0, \tilde{X}^i, t_{f1}^i, t_{f2}^i, t_{f3}^i, t_{f4}^i)$ as defined above. The variable *Cost_sel* (specified in *Hi_des.m*) can be used to redefine the cost $f_{1,des}^i(C_0, \tilde{X}^i, t_{f1}^i)$ in terms of the eigenvalues of the i^{th} ARI

similar to (E.19) and (E.20). For $Cost_sel = 1$ the trace-type cost function is applied, for $Cost_sel = 2$ or $Cost_sel = 3$ the cost function is defined in terms of the eigenvalues of the i^{th} ARI (Note that $Cost_sel$ is a global variable and it is defined in `Hi_des.m`). The specific cost functions $f_{1,des}^i(C_0, \tilde{X}^i, t_{f1}^i)$ corresponding to $Cost_sel$ are as follows.

- $Cost_sel=1$:

$$f_{1,des}^i(C_0, \tilde{X}^i, t_{f1}^i) = Trace \{ e^{ARI^i(C_0, \tilde{X}^i) t_{f1}^i} \}. \quad (E.53)$$

- $Cost_sel=2$:

$$f_{1,des}^i(C_0, \tilde{X}^i, t_{f1}^i) = \sum_{k=1}^{n_{x^i}+n_c} (f_{1,des}^i)_k(C_0, \tilde{X}^i, t_{f1}^i) \quad (E.54)$$

$$(f_{1,des}^i)_k(C_0, \tilde{X}^i, t_{f1}^i) = \begin{cases} [e^{(\lambda_{re})_k^i t_{f1}^i} - 1]^2 & \text{if } (\lambda_{re})_k^i \geq 0 \\ 0 & \text{if } (\lambda_{re})_k^i < 0. \end{cases} \quad (E.55)$$

- $Cost_sel=3$:

$$f_{1,des}^i(C_0, \tilde{X}^i, t_{f1}^i) = \sum_{k=1}^{n_{x^i}+n_c} (f_{1,des}^i)_k(C_0, \tilde{X}^i, t_{f1}^i) \quad (E.56)$$

$$(f_{1,des}^i)_k(C_0, \tilde{X}^i, t_{f1}^i) = \begin{cases} t_{f1}^i [(\lambda_{re})_k^i]^2 & \text{if } (\lambda_{re})_k^i \geq 0 \\ 0 & \text{if } (\lambda_{re})_k^i < 0. \end{cases} \quad (E.57)$$

For $Cost_sel = 2$ and $Cost_sel = 3$ in the above formulae $(\lambda_{re})_k^i$ represents the real part of the k^{th} eigenvalue of the i^{th} ARI: $ARI^i(C_0, \tilde{X}^i)$. The gradients are computed according to the variable $Cost_sel$.

4. **Hi_des_grad.m**

This function computes the gradient of the function $\sum_{i=1}^{n_p} J_{des}^i(C_0, \tilde{X}^i, t_{f1}^i, t_{f2}^i, t_{f3}^i, t_{f4}^i)$ as defined above.

5. **Hi_des_upda.m**

An auxiliary function that updates important variables needed to compute new values for t_{fk}^i .

6. Hi_des_tfup.m

An auxiliary function that computes the new values for t_{fk}^i . The specific values used for the t_{fk}^i -updates have been derived empirically and worked for a range of problems. However, there may be problems where this update scheme is not sufficient. Users who are familiar with the MATLAB script are encouraged to alter this routine for improved convergence. However, when the function is altered the following numerical issues should be examined carefully. First, continuity and smoothness are relative properties when numerical methods are applied to an optimization problem. In general the scaling variables t_{fk}^i should not be increased to large values (in comparison to the other t_{fk}^i values) even if the corresponding inequality constraint is satisfied. Such a t_{fk}^i update scheme would introduce unnecessary numerical problems due to “almost non-smooth” components in the overall cost function. On the other hand, in some cases the values for t_{fk}^i may be “too small” resulting in convergence problems due to large differences (on the order of magnitude) between the gradients with respect to C_0 and those for \tilde{X}^i .

7. Hi_des_opti.m

This file contains the optimizer which is a modified version of the original constrained optimization program `constr.m` (see MATLAB optimization toolbox). The input parameters can be chosen such that information about the progress of the optimization is printed after each line search. In the following a typical sequence for one of the optimizations is displayed.

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	1101.55	0	1	
31	1101.55	0	1.86e-09	
55	1101.25	0	1.19e-07	
:	:	:	:	
:	:	:	:	
385	424.405	0	0.125	mod Hess(2)

Intermediate Optimization Terminated Successfully

In this display “f-COUNT” shows the number of function evaluations used in this particular optimization. The factor “FUNCTION” is the (unscaled) func-

tion value of the overall cost function. This value give hints to how fast the algorithm is converging. If (for $Cost_{sel} = 1$) all relevant scaling factors t_{fk}^i are larger than one and the overall cost function is smaller than one, then the design goal has been reached. However, the \mathcal{H}_∞ -bounds may also be satisfied even when the cost is larger than one. The value “MAX{g}” shows whether or not additional constraints are violated or not and should be zero at all times. The only constraints incorporated in this software are possible bounds on the controller entries. Hence, as long as $MAX\{g\} = 0$ all specified controller constraints are satisfied. The step-size parameter “STEP” is a good indicator as to whether the optimization has converged after the optimization is completed. Extremely small step size parameters indicate that the optimization does not progress very well. The column under “Procedures” displays the mode of the Hessian update.

The individual optimizations started with `Hi_des_opti.m` can terminate with the following diagnostic messages:

- “Intermediate Optimization Terminated Successfully”: This run has converged, the (norm of the) vector for the search direction and the function decrease are sufficiently small.
- “Maximum number of iterations exceeded”: `Hi_des_opti.m` terminates because the number of function evaluations has exceeded the limits specified in the variables `Nr_improveC` and `Nr_improveX`. Clearly in this case convergence has not yet been achieved.
- “Insufficient cost function decrease”: `Hi_des_opti.m` terminates if the function decrease was smaller than 10^{-6} over 6 consecutive line searches with $STEP = 1$. (Most likely the problem requires an update of the t_{fk}^i or it needs a rescaling of the optimization variables).
- “Overall cost is smaller than 1: All Hi-bounds satisfied”: If $Cost_{sel} = 1$ has been selected in `Hi_des.m`, then a function value smaller than one with all $t_{fk}^i > 1$ guarantees that all the specified \mathcal{H}_∞ -bounds are satisfied and a further optimization is not required.
- “Warning: No feasible solution found”: No solution has been found that satisfies the specified upper and lower limits P_U and P_L .

Note that these messages relate to the success of individual optimizations and not to the achievement of the overall \mathcal{H}_∞ -design procedure. Other than these messages the subroutine `Hi_des_qupr.m` may also produce warning messages if the corresponding quadratic programming problem is ill-conditioned or encounters problems. In general the behavior of `Hi_des_opti.m` can be manipulated with the variable “Options” defined in `Hi_des.m`. The Options-vector in `Hi_des.m` has the same meaning as the OPTIONS-vector defined in MATLAB (for more information on this variable, type “help foptions” in MATLAB).

8. **Hi_des_qupr.m**

Quadratic programming subroutine for the computation of the search direction after a line search has been completed.

9. **Hinorm_comp.m**

An auxiliary function to compute the \mathcal{H}_∞ -norm of a linear time-invariant system.

In general extensive information is also included within the function files themselves, and they are rather self-explanatory. Furthermore, all files contain auxiliary warnings and messages detailing the progress and flow of the overall \mathcal{H}_∞ -design procedure.

E.4.3 The Function `Hi_des.m`

As mentioned above, `Hi_des.m` is the main function containing the structural setup of the algorithm as described earlier. This routine can roughly be separated into 6 sections:

1. Input Variable Check

In this part all the input variables are checked for compatibility of the data. The only thing the user has to take care of is to make sure that the system assumptions 1 and 2 as defined in the previous section are satisfied.

2. Generation of Initial Guesses for the \tilde{X}^i

Depending on whether the user hands over an initial guess for these matrices or not, and whether the initial controller guess Co_{in} is stabilizing or not, initial guess for the matrices \tilde{X}^i will be generated (see below).

3. Find a Stabilizing Controller C_0

If the initial controller guess is not stabilizing for all the plant conditions, an iteration is started to try to find such a stabilizing controller. If no stabilizing controller can be found after a number of iterations (please refer to the variables *Nr_iter_stab* and *Func_stabilize* in *Hi_des.m*), then the program will terminate at this point.

4. Improve the Initial Parameters \tilde{X}^i and Co_{in}

If no \tilde{X}_{in} has been defined by the user, or if some of the eigenvalues of the ARI's or the matrices R^i are very large, then a certain number of "improvement loops" is invoked. In these loops a certain number of optimizations (please refer to the variables *Nr_iter_impC*, *Nr_improveC*, *Nr_iter_impX* and *Nr_improveX* in *Hi_des.m*) are performed individually on Co_{in} and the matrices \tilde{X}^i respectively. Failure of convergence can be observed when these values are chosen too small in relation to the number of optimization variables, if one or more of the \mathcal{H}_∞ -bounds γ^i are chosen too small or if the chosen controller structure is too restrictive.

5. Main-Iteration Loop

In this part of the program we optimize on C_0 and the matrices \tilde{X}^i . After each iteration the user will get an update displaying the relevant time t_{f1}^i , the maximum eigenvalue of all ARI's (over all plant conditions) and the achieved \mathcal{H}_∞ -norms for each plant condition.

6. Re-incorporation of a Possible $D_{11} \neq 0$

In this section of the algorithm we re-incorporate the possible case where $D_{11} \neq 0$ (as described before) when the final controller is well-posed, i.e., if $Z = (I - D_{11}D_c^*)^{-1}$ exists.

The function `Hi_des.m` has to be called with 7 input variables and 3 output variables as

$$[Co_{out}, Gamma_{ach}, \tilde{X}_{out}] = \text{Hi_des}(File_name, Gamma, Co_{in}, P_L, P_U, \tilde{X}_{in}, Prnt_{var}).$$

In the following subsection we will discuss the form and meaning of these variables.

E.4.4 Input/Output Arguments of `Hi_des.m`

1. **File_name** (string)

This variable identifies the file where the data was stored during an input session with the function `Hi_des_inpu.m`. The extension *_hi.mat* will be automatically appended to this filename and must not be included in this input argument. For example, *File_name = 'test'* would result in a data file named *'test_hi.mat'* to be loaded.

2. **Gamma, Gamma_{ach}** (vectors)

Gamma (input) contains the desired \mathcal{H}_∞ -bounds for the n_p plant conditions and hence has the dimension $n_p \times 1$ (or $1 \times n_p$).

Gamma_{ach} (output) contains the values of the achieved \mathcal{H}_∞ -norms for the n_p plant conditions after `Hi_des.m` has terminated. *Gamma_{ach}* has the same dimensions as *Gamma*.

3. **Co_{in}, Co_{out}** (matrices)

Co_{in} (input) is the initial controller guess in the format

$$Co_{in} = \begin{pmatrix} D_{c_{in}} & C_{c_{in}} \\ B_{c_{in}} & A_{c_{in}} \end{pmatrix}$$

compatible with C_0 as defined in the previous section and hence has the dimensions $(n_u + n_c) \times (n_y + n_c)$. The controller dimension n_c is determined automatically based on the variables n_u and n_y . In this way the user can choose different controllers with different n_c for the same plant data without having to repeat the input routine `Hi_des_inpu.m`.

Co_{out} (output) contains the final controller after the execution of `Hi_des.m` has been completed.

4. \mathbf{P}_L , \mathbf{P}_U (matrices)

Both of these matrices must be of the same dimensions as the chosen controller Co_{in} . These matrices determine lower and upper bounds on the individual entries of the desired controller, i.e. $P_L \leq Co_{out} \leq P_U$. The bounds P_L and P_U can also be used to fix the structure of the controller. For example, let us assume that $n_u = n_y = 1$ and the chosen input parameters Co_{in} , P_L and P_U are as follows:

$$P_L = \begin{pmatrix} 0 & -10^9 & 0 \\ 0 & 0 & 1 \\ -10^9 & -10^9 & -10^9 \end{pmatrix}, Co_{in} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix}, P_U = \begin{pmatrix} 0 & 10^9 & 100 \\ 0 & 0 & 1 \\ 10^9 & 10^9 & 10^9 \end{pmatrix}.$$

Then the desired controller structure is a strictly proper controller in the controllable canonical form with

$$\begin{aligned} A_c &= \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix} \\ B_c &= \begin{pmatrix} 0 \\ b_{21} \end{pmatrix} \\ C_c &= \begin{pmatrix} c_{11} & c_{12} \end{pmatrix} \\ D_c &= 0 \end{aligned}$$

and the additional constraint $0 \leq c_{12} \leq 100$. The initial controller guess is expected to satisfy the desired bound $P_L \leq Co_{in} \leq P_U$. When the structure of the controller is unconstrained; that is, when we adopt a general proper controller with no bounds on its entries, then the program will assume the default bounds of $-10^9 \leq Co_{ij} \leq 10^9$. In this case `Hi_des.m` can be called with the following arguments:

i) No lower and upper bounds are specified (unconstrained):

$$[Co_{out}, Gamma_{ach}, \tilde{X}_{out}] = \text{Hi_des}(File_name, Gamma, Co_{in}, [], [], \tilde{X}_{in}, Prnt_{var}).$$

ii) Only the upper bounds are specified:

$$[Co_{out}, Gamma_{ach}, \tilde{X}_{out}] = \text{Hi_des}(File_name, Gamma, Co_{in}, [], P_U, \tilde{X}_{in}, Prnt_{var})$$

iii) Only the lower bounds are specified:

$$[Co_{out}, Gamma_{ach}, \tilde{X}_{out}] = \text{Hi_des}(File_name, Gamma, Co_{in}, P_L, [], \tilde{X}_{in}, Prnt_{var})$$

5. $\tilde{\mathbf{X}}_{in}$, $\tilde{\mathbf{X}}_{out}$ (vectors)

As defined in section 3, we optimize over a set of n_p upper triangular matrices \tilde{X}^i . Each of these matrices is of the form

$$\tilde{X}^i = \begin{pmatrix} \tilde{X}_{1,1}^i & \tilde{X}_{1,2}^i & \tilde{X}_{1,3}^i & \cdots & \tilde{X}_{1,(n_{x^i}+n_c-1)}^i & \tilde{X}_{1,(n_{x^i}+n_c)}^i \\ 0 & \tilde{X}_{2,2}^i & \tilde{X}_{2,3}^i & \cdots & \tilde{X}_{2,(n_{x^i}+n_c-1)}^i & \tilde{X}_{2,(n_{x^i}+n_c)}^i \\ 0 & 0 & \tilde{X}_{3,3}^i & \cdots & \tilde{X}_{3,(n_{x^i}+n_c-1)}^i & \tilde{X}_{3,(n_{x^i}+n_c)}^i \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \tilde{X}_{(n_{x^i}+n_c-1),(n_{x^i}+n_c-1)}^i & \tilde{X}_{(n_{x^i}+n_c-1),(n_{x^i}+n_c)}^i \\ 0 & 0 & 0 & \cdots & 0 & \tilde{X}_{(n_{x^i}+n_c),(n_{x^i}+n_c)}^i \end{pmatrix}$$

The variables \tilde{X}_{in} and \tilde{X}_{out} are saved in a vector array as follows

$$\begin{aligned} \tilde{X}_{in} = [& \tilde{X}_{1,1}^1 \tilde{X}_{1,2}^1 \cdots \tilde{X}_{1,(n_{x^1}+n_c)}^1 \cdots \\ & \tilde{X}_{2,2}^1 \tilde{X}_{2,3}^1 \cdots \tilde{X}_{2,(n_{x^1}+n_c)}^1 \cdots \tilde{X}_{(n_{x^1}+n_c),(n_{x^1}+n_c)}^1 \\ & \tilde{X}_{1,1}^2 \tilde{X}_{1,2}^2 \cdots \tilde{X}_{1,(n_{x^2}+n_c)}^2 \cdots \\ & \tilde{X}_{2,2}^2 \tilde{X}_{2,3}^2 \cdots \tilde{X}_{2,(n_{x^2}+n_c)}^2 \cdots \tilde{X}_{(n_{x^2}+n_c),(n_{x^2}+n_c)}^2 \\ & \vdots \\ & \tilde{X}_{1,1}^{n_p} \tilde{X}_{1,2}^{n_p} \cdots \tilde{X}_{1,(n_{x^{n_p}}+n_c)}^{n_p} \cdots \\ & \tilde{X}_{2,2}^{n_p} \tilde{X}_{2,3}^{n_p} \cdots \tilde{X}_{2,(n_{x^{n_p}}+n_c)}^{n_p} \cdots \tilde{X}_{(n_{x^{n_p}}+n_c),(n_{x^{n_p}}+n_c)}^{n_p}] \end{aligned}$$

It is important that Hi_des.m be started with an initial guess \tilde{X}_{in} that satisfies $\tilde{X}^i \geq 0$ (initial guesses with eigenvalues $\lambda(\tilde{X}^i) \leq -0.2$ are rejected and result in an error message!). Typically when the initial guess of \tilde{X}_{in} is not available, then Hi_des.m may be invoked with the following input arguments

$$[Co_{out}, Gamma_{ach}, \tilde{X}_{out}] = \text{Hi_des}(File_name, Gamma, Co_{in}, P_L, P_U, [], Prnt_{var}).$$

Note that one could also restart Hi_des.m with \tilde{X}_{out} obtained from an earlier run as initial guess for \tilde{X}_{in} . In this case, the above required conditions will be automatically satisfied.

6. **Prnt_{var}** (non-negative scalar) This input variable controls the amount of screen output during the optimization.

- $0 \leq Prnt_{var} < 5$: Only the most important messages are printed.
- $5 \leq Prnt_{var} < 10$: Only the most important diagnostic messages and a brief summary of the design results after each main-loop iteration are printed.
- $10 \leq Prnt_{var}$: In addition to the above output `Hi_des_opti.m` will print the function value, the step size and the number of function evaluations after each line search.

E.5 Examples

E.5.1 Example 1: One-Plant Case

This example is taken from [88] (see also example 1 in chapter 5). It is a 3rd-order single input/single output plant with one measurement and one control input. The example satisfies the system assumptions in [24]. For more details on this plant please refer to [88]. In the following we will first demonstrate how to enter the data using the input routine `Hi_des_inpu.m`. Then we show excerpts from the diary file created during the `Hi_des.m` execution with this plant data.

E.5.2 Input of the Plant Data: `Hi_des_inpu.m`

Before the `Hi_des_inpu.m` file is called, a MATLAB data file *Rid.mat* must be created in which the system matrix has been stored under the variable name *S1*. This simplifies the input of the plant data in the routine `Hi_des_inpu.m` considerably. The actual call of `Hi_des_inpu.m` produces the following diary file.

```
>> Hi_des_inpu

Are you a new user or need preliminary information? -- y/n    n
*****

Input the file name (char) under which the plant parameters will be
stored.

Please enter the input file name :   Example1

NOTE: If the file Example1_hi.mat already exists, it will be overridden !

*****
```

```

Input the number Np (integer>=1) of different plant conditions :1
*****

Input the number Ny of measurement outputs : 1
*****

Input the number Nu of control inputs : 1
*****

PLANT CONDITION 1 :

Enter the dimension vector [ Nx , Nwi , Nzi ] : [3 1 1]
*****

Input the corresponding system matrices for
PLANT CONDITION 1 with the following format:

      [A    , B_1  , B_2  ;
      C_1  , D_11  , D_12  ;
      C_2  , D_21  , D_22  ]

-- Pressing the <RETURN> key will put you into the keyboard
mode to load data. When done, enter the command
"return" to exit the keyboard mode and continue the program.

Enter the system matrix (e.g.,[a,b;c,d]):<RETURN>
K>> load Rid
K>> return

Enter the system matrix (e.g.,[a,b;c,d]): S1
*****

Please wait.... Saving the data in Example1_hi.mat
Done

```

While waiting for the the input of the system matrix, the <RETURN> key is pressed. This places MATLAB into the keyboard mode. The file *Rid.mat* containing the system matrix under the variable name S1 is then loaded. We exit the keyboard mode to the normal mode by entering “return”. We give the system matrix S1 as the

input data. This completes the input of the plant data. The file “Example1_hi.mat” is now the basis for the subsequent call to Hi_des.m.

E.5.3 Controller Design: Hi_des.m

To illustrate the working of the Hi_des.m m-file for the H_∞ control problem, we include below a listing of the file Example1.m which contains a setup of the initial design variables and a call to Hi_des.m.

```
%
% Example1.m file
%
clear
clear global

File_name = 'Example1'
eval(['load ' File_name '_hi'])
Global_var
System_1
Dimensions_1
n_p
n_u
n_y
pause

Gamma      = 2.3;
Co_in      = [0      1      2      3;
               1      0      1      0;
               1      0      0      1;
               1      1      2      3];

P_L        = 1e9*[      0    -1.0000    -1.0000    -1.0000;
                   -1.0000    -1.0000    -1.0000    -1.0000;
                   -1.0000    -1.0000    -1.0000    -1.0000;
                   -1.0000    -1.0000    -1.0000    -1.0000];

P_U        = 1e9*[      0     1.0000     1.0000     1.0000;
                   1.0000     1.0000     1.0000     1.0000;
                   1.0000     1.0000     1.0000     1.0000;
                   1.0000     1.0000     1.0000     1.0000];

X_in       = [];
```

```

Prnt_var = 30;

[Co_out,Gamma_ach,X_out] = ...
    Hi_des(File_name,Gamma,Co_in,P_L,P_U,X_in,Prnt_var)
%
% End Example1.m
%
```

The design goal is to synthesize a full-order ($n_c = 3$) strictly proper controller that satisfies the specified \mathcal{H}_∞ -constraint $\Gamma = 2.3$. As reported in [88], the minimally achievable \mathcal{H}_∞ -norm for this plant is approximately 2.1426. For the following sample run the following parameter values were chosen: $Nr_iter_impC = 4$, $Nr_improveC = 400$, $Nr_iter_impX = 3$, $Nr_improveX = 700$ and $Cost_sel = 1$. Execution of Example1.m produces design results that are saved in a diary file. A partial listing of the results is given below.

```

>>Example1

File_name =
Example1

Global_var =
    Dimensions_1 System_1

System_1 =
    -0.3908    -0.4565     1.2657    -0.4275     0.0488
     1.4453    -1.0491    -1.2077    -0.4470     0.3608
    -0.1288     0.6744     1.0324    -0.9172     0.3564
    -1.5567    -1.9432    -0.0914         0     0.5185
     0.9420     0.0144     0.1187     1.3575         0

Dimensions_1 =
     3     1     1

n_p =
     1

n_u =
     1

n_y =
     1
```


Gamma =

2.3

Co_in =

0	1	2	3
1	0	1	0
1	0	0	1
1	1	2	3

P_L =

1.0e+09 *

0	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1

P_U =

1.0e+09 *

0	1	1	1
1	1	1	1
1	1	1	1
1	1	1	1

X_in =

[]

Prnt_var =

30

Maximum real part of all A_cl eigenvalues : Lambda_max_Acl = 5.42

Trying to find stabilizing controller satisfying

sigma_max(D_cl) < Gamma for all plant conditions!

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	27.115	0	1	
2	1.6919	0	1	
3	0.0337288	0	1	
:	:	:	:	
18	8.73314e-24	0	1	mod Hess

Intermediate Optimization Terminated Successfully

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
---------	----------	--------	------	------------

1	9.47196e-15	0	1	
2	0	0	1	mod Hess
3	0	0	1	mod Hess

Intermediate Optimization Terminated Successfully

The controller is stabilizing all plant conditions
and satisfies $\sigma_{\max}(D_{\text{cli}}) < \gamma_i$!
for all plant conditions.
Continuing to improvement phase/main iteration loop

No initial guess for X_{in} is given!

OR:

Some of the ARI eigenvalues are extremely large
Trying to find better initial guesses for the main iteration
- or as initial guesses for a restart of this program

Starting the improvement phase:

Optimizing on C_0 only:

Updating Tf-values, please wait

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	8.23622	0	1	
2	6.00063	0	1	
3	6.00063	0	1	

Intermediate Optimization Terminated Successfully

Updating Tf-values, please wait

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	41.0173	0	1	
2	6.00477	0	1	
3	6.00477	0	1	mod Hess
:	:	:	:	
17	6.00046	0	1	mod Hess

Insufficient cost function decrease

Updating Tf-values, please wait

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	1101.63	0	1	
14	858.956	0	0.000244	
27	109.521	0	0.000244	
:	:	:	:	
399	6.02436	0	1	mod Hess

Maximum number of iterations exceeded

Updating Tf-values, please wait

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	1101.55	0	1	
31	1101.55	0	1.86e-09	
55	1101.25	0	1.19e-07	
:	:	:	:	
385	424.405	0	0.125	mod Hess(2)

Intermediate Optimization Terminated Successfully

Number of improvement steps on Co exceeded,
continuing with optimization on X only:

Updating Tf-values, please wait

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	48.1392	0	1	
12	44.2766	0	0.000977	
25	42.7976	0	0.000244	
:	:	:	:	
643	5.09314	0	1	mod Hess

Insufficient cost function decrease

Updating Tf-values, please wait

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
---------	----------	--------	------	------------

1	426.689	0	1	
20	426.098	0	3.81e-06	
36	413.426	0	3.05e-05	
:	:	:	:	
423	5.53867	0	1	mod Hess

Insufficient cost function decrease

Updating Tf-values, please wait

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	59.9748	0	1	
26	59.9744	0	5.96e-08	
49	59.9559	0	2.38e-07	
:	:	:	:	
699	34.7229	0	1	
Maximum number of iterations exceeded				

End of improvement phase,
Continuing with main iteration loop

Updating Tf-values, please wait

```

Starting Iteration      1 with (minimal)           Tf =  87.4
Maximum real part of all ARI eigenvalues :      Lambda_max_ARI = 0.0366
Maximum real part of all A_cl eigenvalues:      Lambda_max_Acl = -0.787
Achieved H_inf-norms for all plants:
                                System      1 : 2.86993

```

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	35.6239	0	1	
30	35.6238	0	3.73e-09	
56	35.6223	0	2.98e-08	
:	:	:	:	
886	2.93837	0	1.49e-08	
Intermediate Optimization Terminated Successfully				

All Hi-constraints satisfied

Co_out =

```

1.0e+02 *
      0  0.00267772193123  0.00091843489067  0.00144300504148
-5.61830215774721 -0.52007701815391 -0.03497247440197 -0.40920200207161
-2.49550980375480 -0.60157220605899 -0.43420296505282  0.60648866763827
-5.38689628028888 -0.72850304408728 -0.29185379848967  0.09084576068794

```

Gamma_ach =

2.23701497229569

X_out =

```

1.0e+03 *
Columns 1 through 4
1.15630229860465  0.09249883566253 -0.67301817161558 -0.19397056072923

Columns 5 through 8
-0.17205624462823  0.28165410889259  0.70696049742409 -0.15953506186834

Columns 9 through 12
0.05674695169626  0.05866233452346 -0.08565256194329  0.40848499439910

Columns 13 through 16
0.10204208348404  0.08925625378226 -0.14769868862166  0.04001880566657

Columns 17 through 20
0.03635819909302 -0.05844568069746  0.03311347764003 -0.05313227809173

Column 21
0.08537398653844

```

As it can be seen, the algorithm converges to a controller that satisfies the specified \mathcal{H}_∞ -bound. After the improvement phase, only one main iteration was necessary to satisfy the design goal. This fact further illustrates the importance of good initial guesses for the controller Co_{in} and for the matrices \tilde{X}^i . If no such good initial guesses were available, then appropriate starting values will be determined in the preliminary optimization phase. The improvement phase resulted in a maximum ARI-eigenvalue of 0.0366. During the subsequent main iterations the \mathcal{H}_∞ -norm was reduced from

$\Gamma_{ach} = 2.86993$ to $\Gamma_{ach} = 2.23701$.

E.5.4 Example 2: Non-Standard Plant

The example considered does not have any particular physical meaning. However, it is a rather challenging problem since it violates all the system assumptions made in [24] and [113] hence rendering these approaches incapable to solve this problem. The open-loop system $\Sigma_{\infty,op}$ is given as follows.

$$\begin{aligned}\dot{\bar{x}}^1 &= \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \bar{x}^1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u^1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w_{\infty}^1 \\ y^1 &= \begin{pmatrix} 0 & 1 \end{pmatrix} \bar{x}^1 \\ z_{\infty}^1 &= \begin{pmatrix} -1 & 1 \end{pmatrix} \bar{x}^1\end{aligned}$$

This plant is detectable through y^1 (but unobservable) and stabilizable via u^1 (but uncontrollable). Furthermore, the problem is singular and certain subsystems have invariant zeros on the $j\omega$ -axis (please refer to [24] and [113]). The open-loop system is unstable and the controller for this example is a static output-feedback controller of the form $u = Ky$. It is easily verified that the controller will stabilize the plant for any $K < -1$. The \mathcal{H}_{∞} -norm of the closed-loop transfer function from w_{∞}^1 to z_{∞}^1 is given by

$$\|T_{\infty}(K, s)\|_{\infty} = \frac{1}{|K|}$$

The startup file Example2.m for this case is:

```
%
% Example2.m file
%
clear
clear global

File_name = 'Example2'
eval(['load ' File_name '_hi'])
Global_var
System_1
Dimensions_1
n_p
n_u
n_y
```

```

pause

Gamma      = 1e-4;
Co_in      = 0;
P_L        = -1e5;
P_U        = 0;
X_in       = [];
Prnt_var   = 30;

[Co_out,Gamma_ach,X_out] = ...
    Hi_des(File_name,Gamma,Co_in,P_L,P_U,X_in,Prnt_var)
%
% End Example2.m
%
```

Here we start out with a destabilizing controller and allow K to vary within the bounds $-10^5 \leq K \leq 0$. To achieve the desired \mathcal{H}_∞ -bound of $\Gamma = 10^{-4}$ the algorithm has to find a controller $K < -10^4$. The following diary shows the execution of the file Example2.m.

```

>> Example2

File_name =

Example2

Global_var =
    Dimensions_1 System_1

System_1 =
    -1     1     0     0
     0     1     1     1
     0     1     0     0
    -1     1     0     0

Dimensions_1 =
     2     1     1

n_p =
     1

n_u =
     1
```

240

n_y =
1

Gamma =
1.0000000000000000e-04

Co_in =
0

P_L =
-100000

P_U =
0

X_in =
[]

Prnt_var =
30

Maximum real part of all A_cl eigenvalues : Lambda_max_Acl = 1
Trying to find stabilizing controller satisfying
sigma_max(D_cl) < Gamma for all plant conditions!

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	5.99999	0	1	
2	0.999988	0	1	
3	0.999988	0	1	

Intermediate Optimization Terminated Successfully

The controller is stabilizing all plant conditions
and satisfies sigma_max(D_cli) < gamma_i !
for all plant conditions.
Continuing to improvement phase/main iteration loop

No initial guess for X_in is given!

OR:

Some of the ARI eigenvalues are extremely large

Trying to find better initial guesses for the main iteration

- or as initial guesses for a restart of this program

Starting the improvement phase:
Optimizing on Co only:

Updating Tf-values, please wait

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	1098.81	0	1	
2	3.02448	0	1	
3	3.02448	0	1	mod Hess
:	:	:	:	
19	3.00101	0	1	mod Hess
Warning: QP problem is -ve semi-definite.				
20	3.00033	0	1	mod Hess
Warning: QP problem is -ve semi-definite.				
21	2.99863	0	1	mod Hess
22	2.99863	0	1	mod Hess

Intermediate Optimization Terminated Successfully

All Hi-constraints satisfied,
terminating Co-improvements;
Attempting to find solutions X for the ARIs

Number of improvement steps on Co exceeded,
continuing with optimization on X only:

Updating Tf-values, please wait

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	3.00775	0	1	
6	2.81566	0	0.0625	
8	2.5653	0	0.5	

Overall cost is smaller than 1: All Hi-bounds satisfied

All Hi-constraints have been satisfied after the improvement phase

```
Co_out =
    -1.0000000000000000e+05
```

```
Gamma_ach =
    1.000000002370170e-05
```

```
X_out =
    0.17528500753256    0.00000102832300    0.00094205358802
```

The algorithm found the desired controller already in the “improvement phase” of the algorithm. In the subsequent optimization the algorithm attempts to find a solution \tilde{X} that satisfies the corresponding ARI-constraint. The Quadratic Programming sub-problem during the optimization became semi-definite as the controller gain K reached its specified limit $P_U = -10^5$.

This example shows that this algorithm can accomodate a much larger class of H_∞ control problems and systems than the approaches in [24] and [113].

E.5.5 Example 3: Multiple Plant Case

The last example is a simultaneous \mathcal{H}_∞ -design for a F-15 aircraft at two operating conditions. This example has been investigated in [105]. The operating conditions represent a subsonic and a supersonic flight condition. Both models are of 4th-order and the controller to be designed is a proper 1st-order controller. The corresponding startup file is in Example3.m and its listing is given below.

```
%
% Example3.m file
%
clear
clear global
File_name = 'Example3'
eval(['load ' File_name '_hi'])
Global_var
System_1
System_2
Dimensions_1
Dimensions_2
```

```

n_p
n_u
n_y
pause

Gamma      = [0.11 0.15];

Co_in      =      [ 1   2   3   4   1;
                   1   2   3   4   0];

P_L        = 1e9*[-1   -1   -1   -1   -1;
                 -1   -1   -1   -1   -1];

P_U        = 1e9*[ 1   1   1   1   1;
                 1   1   1   1   1];

X_in       = [];

Prnt_var   = 6;
[Co_out,Gamma_ach,X_out] = ...
    Hi_des(File_name,Gamma,Co_in,P_L,P_U,X_in,Prnt_var)
%
% End Example3.m
%
```

The first execution of Example3.m resulted in the following output results.

```
>> Example3
```

```
File_name =
Example3
```

```
Global_var =
Dimensions_1 System_1 Dimensions_2 System_2
```

```
System_1 =
Columns 1 through 8
```

```

-0.0082 -25.70      0 -32.17  -6.80  -0.55      0      0
-0.0002  -1.27    1.00      0  -0.14      0  -0.27  20.99
 0.0007   1.02  -2.42      0 -14.06      0   0.99      0
      0      0    1.00      0      0      0      0      0
 1.0000      0      0      0      0      0      0      0
      0   1.00      0      0      0      0      0      0
```

0	0	1.00	0	0	0	0	0
0	0	0	1.00	0	0	0	0
0.0147	0	0	0	0	0	0	0
0	0.01	0	0	0	0	0	0
0	-0.16	0	0	0	0	0	0

System_2 =
Columns 1 through 8

-0.01	-95.91	0	-32.11	-25.40	-0.79	0	0
-0.0001	-1.87	1.00	0	-0.22	0	-0.39	0
0.0006	-3.61	-3.44	0	-53.42	0	2.04	78.46
0	0	1.00	0	0	0	0	0
1.0000	0	0	0	0	0	0	0
0	1.00	0	0	0	0	0	0
0	0	1.00	0	0	0	0	0
0	0	0	1.00	0	0	0	0
0.0147	0	0	0	0	0	0	0
0	0.01	0	0	0	0	0	0
0	-0.16	0	0	0	0	0	0

Dimensions_1 =
4 3 3

Dimensions_2 =
4 3 3

n_p =
2

n_u =
1

n_y =
4

Gamma =
0.1000 0.1500

Co_in =
1 2 3 4 1
1 2 3 4 0

P_L =

```

1.0e+09 *
-1.0000  -1.0000  -1.0000  -1.0000  -1.0000
-1.0000  -1.0000  -1.0000  -1.0000  -1.0000

P_U =
1.0e+09 *
1.0000  1.0000  1.0000  1.0000  1.0000
1.0000  1.0000  1.0000  1.0000  1.0000

X_in =
[]

Prnt_var =
30

Maximum real part of all A_cl eigenvalues : Lambda_max_Acl = 4.38
Trying to find stabilizing controller satisfying
sigma_max(D_cl) < Gamma for all plant conditions!

f-COUNT  FUNCTION      MAX{g}      STEP  Procedures
1        36.4748        0           1
5        1.118e-05        0         0.125
6        1.118e-05        0           1
Intermediate Optimization Terminated Successfully

The controller is stabilizing all plant conditions

No initial guess for X_in is given!
OR:
Some of the ARI eigenvalues are extremely large
Trying to find better initial guesses for the main iteration
- or as initial guesses for a restart of this program

Starting the improvement phase:
Optimizing on Co only:

Updating Tf-values, please wait

```

```

f-COUNT  FUNCTION      MAX{g}      STEP  Procedures
1        1748.08        0           1

```

18	1654.85	0	1.53e-05	
35	1062.08	0	1.53e-05	
:	:	:	:	
222	19.642	0	1	mod Hess

Intermediate Optimization Terminated Successfully

Updating Tf-values, please wait

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	472.433	0	1	
19	470.111	0	7.63e-06	
38	457.481	0	3.81e-06	
:	:	:	:	
177	356.439	0	1	mod Hess

Intermediate Optimization Terminated Successfully

Updating Tf-values, please wait

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	522.758	0	1	
23	522.721	0	4.77e-07	
44	522.161	0	9.54e-07	
:	:	:	:	
164	517.261	0	1	mod Hess

Intermediate Optimization Terminated Successfully

Updating Tf-values, please wait

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	545.121	0	1	
26	545.098	0	5.96e-08	
51	545.058	0	5.96e-08	
:	:	:	:	
195	544.987	0	1	mod Hess

Intermediate Optimization Terminated Successfully

Number of improvement steps on Co exceeded,

continuing with optimization on X only:

Updating Tf-values, please wait

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	30.0505	0	1	
14	29.3947	0	0.000244	
29	28.5179	0	6.1e-05	
:	:	:	:	
474	11.0803	0	1	

Intermediate Optimization Terminated Successfully

Updating Tf-values, please wait

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	37.9007	0	1	
28	37.9006	0	1.49e-08	
62	37.9004	0	1.16e-10	
:	:	:	:	
499	9.94873	0	1	mod Hess

Maximum number of iterations exceeded

Updating Tf-values, please wait

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	17.7262	0	1	
25	17.7261	0	1.19e-07	
50	17.7257	0	5.96e-08	
:	:	:	:	
499	15.1179	0	0.125	

Maximum number of iterations exceeded

End of improvement phase,
Continuing with main iteration loop

Updating Tf-values, please wait

Starting Iteration 1 with (minimal) Tf = 750
 Maximum real part of all ARI eigenvalues : Lambda_max_ARI = 0.00267
 Maximum real part of all A_cl eigenvalues: Lambda_max_Acl = -0.445
 Achieved H_{inf}-norms for all plants:
 System 1 : 0.178205
 System 2 : 0.248668

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	16.5223	0	1	
26	16.5222	0	5.96e-08	
54	16.5222	0	7.45e-09	
:	:	:	:	
898	3.11972	0	0.5	mod Hess

Maximum number of iterations exceeded

All Hi-constraints satisfied

Co_out =

1.0e+02 *

Columns 1 through 4

-0.12550105856800	0.89086997906138	0.35143782391798	1.35261095959955
-0.33497386811212	2.41299990656566	0.92063181212225	3.52022469854965

Column 5

0.07546321725795
0.13073533249575

Gamma_ach =

0.05695666274842	0.12963242582712
------------------	------------------

X_out =

Columns 1 through 4

0.00395899934532	-0.01900536171326	-0.00194982498687	-0.01609309367332
------------------	-------------------	-------------------	-------------------

Columns 5 through 8

0.00750824600468	0.22372281620987	0.01491783289404	0.00190207501306
------------------	------------------	------------------	------------------

Columns 9 through 12

-0.01651842028388 0.00604400182495 0.02339670805051 0.00234704648193

Columns 13 through 16

0.28261543024549 -0.00788095393616 0.02817775033799 0.00098758201217

Columns 17 through 20

-0.00716432022466 -0.00207342379310 -0.00643992932888 0.00140348168097

Columns 21 through 24

0.10793274494332 0.01466806585088 0.03045749624873 -0.00189741626634

Columns 25 through 28

0.00573040760352 0.02165813619593 0.00083333222729 0.14515239198804

Columns 29 through 30

0.02716374583356 0.02244058255040

Once again the design goal has been reached after the first main iteration, emphasizing once again the importance of good initial guesses. However, for the same multi-plant problem, when we choose a set of smaller bounds on $\Gamma = [0.1, 0.11]$ and start the design optimization with the same initial guesses as before, clearly a satisfactory convergence will take a slightly longer time. A log of the program execution is given below after the initial improvement phase.

```

Starting Iteration   1 with (minimal)           Tf =   358
Maximum real part of all ARI eigenvalues :      Lambda_max_ARI = 0.00558
Maximum real part of all A_cl eigenvalues:      Lambda_max_Acl = -0.419
Achieved H_inf-norms for all plants:
                                System      1 : 0.163487
                                System      2 : 0.262504

```

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	24.5368	0	1	
31	24.5368	0	1.86e-09	
63	24.5368	0	4.66e-10	
:	:	:	:	
899	3.90574	0	1	mod Hess

Maximum number of iterations exceeded

Updating Tf-values, please wait

Starting Iteration 2 with (minimal) Tf = 2.46e+03
 Maximum real part of all ARI eigenvalues : Lambda_max_ARI = 0.000266
 Maximum real part of all A_cl eigenvalues: Lambda_max_Acl = -0.429
 Achieved H_inf-norms for all plants:
 System 1 : 0.0798264
 System 2 : 0.209862

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	3.99268	0	1	
20	3.99239	0	3.81e-06	
43	3.99203	0	2.38e-07	
:	:	:	:	
899	2.29454	0	0.5	

Maximum number of iterations exceeded

Updating Tf-values, please wait

Starting Iteration 3 with (minimal) Tf = 4.51e+03
 Maximum real part of all ARI eigenvalues : Lambda_max_ARI = 9.3e-05
 Maximum real part of all A_cl eigenvalues: Lambda_max_Acl = -0.445
 Achieved H_inf-norms for all plants:
 System 1 : 0.0603576
 System 2 : 0.136723

f-COUNT	FUNCTION	MAX{g}	STEP	Procedures
1	2.25419	0	1	
22	2.25415	0	9.54e-07	
45	2.25396	0	2.38e-07	
:	:	:	:	
899	1.65992	0	1	mod Hess

Maximum number of iterations exceeded

All Hi-constraints satisfied

Co_out =

1.0e+02 *

Columns 1 through 4

-0.41791228209161	4.44752162549461	0.63314014709939	3.04060895422836
-0.46116010039480	4.96770681319143	0.72541231182832	3.46117417794747

Column 5

-0.08449083071726

-0.13359059043192

Gamma_ach =

0.06582983596218	0.09846974908558
------------------	------------------

X_out =

Columns 1 through 4

0.00954172281255	-0.10435603865224	-0.01003061271832	-0.07181595360680
------------------	-------------------	-------------------	-------------------

Columns 5 through 8

0.00536221146395	1.23228899877861	0.10661647033237	0.70055503142613
------------------	------------------	------------------	------------------

Columns 9 through 12

-0.06444161581800	0.01525554772346	0.07260760853509	-0.00173347068189
-------------------	------------------	------------------	-------------------

Columns 13 through 16

0.79158344370358	-0.05645086683592	0.01048230245748	0.00252476732077
------------------	-------------------	------------------	------------------

Columns 17 through 20

-0.02787365098033	-0.00306582874682	-0.03204022536248	0.00648645844409
-------------------	-------------------	-------------------	------------------

Columns 21 through 24

0.32418777048187	0.03260732184157	0.36598113312310	-0.08052795714074
------------------	------------------	------------------	-------------------

Columns 25 through 28

0.00466389967999	0.02229485722969	-0.00059685472857	0.80777388969307
------------------	------------------	-------------------	------------------

Columns 29 through 30

-0.24917380201575	0.08809115810297
-------------------	------------------

The specified \mathcal{H}_∞ -bounds have been achieved after 3 main iterations. In [105] it

was reported that the minimally achievable \mathcal{H}_∞ -norm for the first plant condition is approximately 0.056 and that for the second plant condition is 0.096. Hence, the specified \mathcal{H}_∞ -bounds in the above problem are very close to the optimally achievable values. As a “rule of thumb” it is expected that the computation time for convergence will increase when the chosen *Gamma*-bounds are getting closer to the minimally achievable \mathcal{H}_∞ -norms and also when the problem size is large. Note also that the maximum eigenvalue of all the ARI’s should be a decreasing function of the iteration number, while the scaling factor “Tf” (the minimum over all scale factors t_{f1}^i) is a monotonically increasing function. This constitutes the expected and desired behavior of the algorithm. Failure of convergence would generally be characterized by “stagnant Tf values”, and the maximum eigenvalue of all the ARI’s is a non-monotonically decreasing function of the iteration number.